Joint Source-Channel Coding for Arbitrarily Varying Wyner-Ziv Source and Gel’fand-Pinsker Channel

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- Joint source-channel coding formulation
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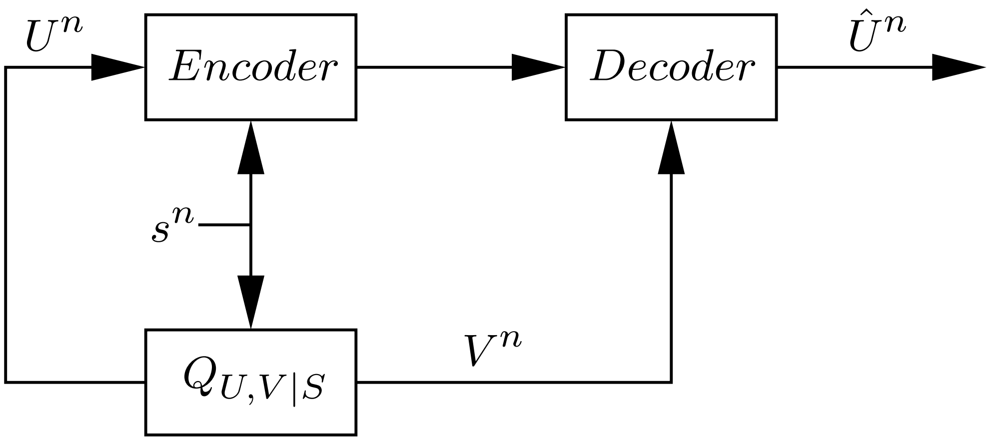
- Joint AV-WZ source and AV-GP channel
- Proof idea

Rate-distortion problem formulation



- $Q_{U,V|S}$ – Memoryless Wyner-Ziv source with state variable s, (AV-WZ).
- s^n - Arbitrarily varying state sequence, known non-causally at the encoder.
- V^n - Decoder side information.

We are interested in the (deterministic) rate-distortion function, $R(D)$.

Rate-distortion problem formulation (cont'd)

Notation and assumptions:

- Finite alphabets $\mathcal{U}$, $\mathcal{V}$, $\mathcal{S}$.
- Memoryless source

$$Q(u^n, v^n | s^n) = \prod_{i=1}^{n} Q(u_i, v_i | s_i)$$

- The distortion between source sequence u^n and reproduction sequence $\hat{u}^n$ is measured by

$$d(u^n, \hat{u}^n) = \frac{1}{n} \sum_{i=1}^{n} d(u_i, \hat{u}_i)$$

  where $\hat{\mathcal{U}}$ is the reproduction alphabet, and $d(u_i, \hat{u}_i)$ is a bounded distortion measure:

$$\max_{u \in \mathcal{U}, \hat{u} \in \hat{\mathcal{U}}} d(u, \hat{u}) \leq D_{\text{max}}.$$

Rate-distortion problem formulation (cont'd)

A $(2^{nR}, n, D)$ deterministic code for the AV-WZ source is a pair of maps (f, ϕ)

$$f : \mathcal{U}^n \times \mathcal{S}^n \to \left\{1, 2, \ldots, 2^{nR}\right\}$$
$$\phi : \left\{1, 2, \ldots, 2^{nR}\right\} \times \mathcal{V}^n \to \mathcal{U}^n$$

such that

$$D \geq \max_{q \in \mathcal{P}(\mathcal{S}^n)} \sum_{s^n, u^n, v^n} q(s^n) Q(u^n, v^n | s^n) d(u^n, \phi(f(u^n, s^n), v^n))$$

The *rate* of the code is R.

A rate R is *achievable* with distortion D if for any $\gamma > 0$ and sufficiently large n, there exists an $(2^{nR}, n, D + \gamma)$ deterministic code for the AV-WZ source.

The *rate-distortion function* $R(D)$ is the infimum over all achievable rates.

Rate-distortion problem formulation (cont'd)

Correlated, randomized-encoder and deterministic rate-distortion codes

- Deterministic rate-distortion code:
  Deterministic encoder: $i = f(u^n, s^n)$
  Deterministic decoder: $\hat{u}^n = \varphi(i, v^n)$
- Average distortion:

$$D(s^n) = \sum_{u^n, v^n} Q(u^n, v^n | s^n) d(u^n, \varphi(f(u^n, s^n), v^n))$$

- Miss represented set:

$$\Delta(s^n) \triangleq \left\{(u^n, v^n) \in \mathcal{U}^n \times \mathcal{V}^n : d(u^n, \varphi(f(u^n, s^n), v^n)) > D + \delta\right\}.$$

- Randomized-encoder rate-distortion code:
  Probabilistic encoder: $P(i | u^n, s^n)$
  Deterministic decoder: $\hat{u}^n = \varphi(i, v^n)$
- Average distortion:

$$D(s^n) = \sum_{u^n, v^n} \sum_i Q(u^n, v^n | s^n) P(i | u^n, s^n) d(u^n, \varphi(i, v^n))$$

Rate-distortion problem formulation (cont'd)

Correlated, randomized-encoder and deterministic rate-distortion codes (cont'd)

- Correlated rate-distortion code:

  Collection of deterministic codes: $\{(f^\gamma, \varphi^\gamma)\}_{\gamma \in \Gamma}$

  Probability distribution on the codes collection Γ: μ

- Average distortion:

$$D(s^n) = \sum_{\gamma} \mu(\gamma) \sum_{u^n, v^v} Q(u^n, v^n | s^n) d(u^n, \varphi^\gamma(f^\gamma(u^n, s^n), v^n)).$$

Previous work

- Wyner & Ziv, 1976 – Source with side information at the decoder

$$R(D) = \min_{\mathcal{M}(D)} [I(Z;U) - I(Z;V)],$$

$\mathcal{M}(D)$ – All RVs $\{U, V, Z, \hat{U}\}$ with probability distribution $Q(U,V)P_{Z|U}P_{\hat{U}|Z,V}$, such that $\mathbb{E}d(U,\hat{U}) \leq D$.

- Gel'fand & Pinsker, 1980 – Capacity formula for channel with random parameters and non-causal CSIT

$$C = \max_{P_{U,X|S}} [I(U;Y) - I(U;S)], \; U \multimap (X,S) \multimap Y.$$

Previous work (cont'd)

- Ahlswede's elimination technique ,1978 – An AVC deterministic code capacity either equals its correlated code capacity or else is zero (in the absence of side information and constraints).
  Only polynomial (in blocklength) large codes collection is needed.
- Ahlswede, 1986 – Positivity and capacity for an AVC with non-causal CSIT
  - Separation Lemma – Deterministic code capacity is positive iff for every state s the DMC $w(y|x,s)$ has positive capacity.
  - If the deterministic code capacity is positive $C = \min_q C^q$.
  - Achievability in three steps:
    1. Extension of Gel'fand & Pinsker to a compound channel
    2. Correlated code capacity equals compound channel capacity
    3. Deterministic code capacity equals correlated code capacity

AV-WZ rate-distortion function

- $Q(U,V|S)$ – AV-WZ source, s^n is known at the encoder.
- $q(\cdot)$ – An arbitrary probability on the state space $\mathcal{S}$.
- $R^q(D)$ – Rate-distortion of $Q(U,V|S)$ with random parameters with probability q (Wyner-Ziv, 1976).

$$R^q(D) = \min[I(Z;S,U) - I(Z;V)]$$

where the minimum is over all $P_{Z|U,S}$ and functions $\varphi : \mathcal{Z} \times \mathcal{V} \to \hat{\mathcal{U}}$ such that

$$\mathbb{E}d(U,\hat{U}) \le D.$$

AV-WZ rate-distortion function – main result.

$$\begin{aligned} R^q(D) &= \min_{P_{Z|U,S},\ \varphi:\mathcal{Z}\times\mathcal{V}\to\hat{\mathcal{U}}} [I(Z;S,U) - I(Z;V)] \\ D &\geq \mathbb{E}d(U,\hat{U}). \end{aligned}$$

<u>Theorem</u>:

The rate-distortion function of the discrete memoryless arbitrarily varying Wyner-Ziv source, with state sequence s^n known at the encoder, is given by:

$$R(D) = \max_q R^q(D)$$

Proof idea

Converse

If $R < \max_q R^q(D)$ is achievable, then there exists q' such that $R < R^{q'}(D)$. Averaging the distortion with respect to q' , results in a contradiction to Wyner-Ziv formula.

Achievability

Achievability of the rate-distortion is shown following Ahlswede 1986 ideas, applied to the AV-WZ setting. Proof in three steps:

1. Rate-distortion for compound (in S) Wyner-Ziv source
2. Construction of correlated code for AV-WZ source (robustification technique)
3. Reduction of correlation (common randomness) between encoder and decoder (elimination technique)

Proof idea (cont'd)

First step: Compound WZ source

- State sequence is chosen i.i.d with unknown probability.
- Key Idea: preassign to every state type t a code for state-dependent WZ source, with state distribution t', with good distortion performance

$$\sum_{\mathcal{S}(t,n)} Q(\Delta^{t'}(s^n)|s^n) \leq \inf_q \sum_{\mathcal{S}(t,n)} Q(\Delta^q(s^n)|s^n) + e^{-\epsilon n}$$

- Inform the decoder on the type by concatenating a type preamble to the code.

Here $\Delta^q(s^n)$ stands for the miss represented set of a code (f, ϕ), that achieves (R, D) for the WZ source with state distribution q.

Proof idea (cont'd)

First step: Compound WZ source (cont'd)

- The receiver selects the appropriate decoder according to the preamble.
- The number of different types is only polynomial in blocklength.

  Does not affect the overall rate.
- Miss represented set has exponentially small probability:

  Deterministic codes (can be shown to) have miss represented sets with exponentially small probability.

  The preamble has exponentially small error probability.

Proof idea (cont'd)

Second step: Construction of a correlated code using the RT

Robustification Technique (Ahlswede, 1986)

- $g : s^n \to [0, 1]$
- π – a permutation function on $\mathcal{S}^n$

If for every memoryless probability p^n:

$$\sum_{s^n} p^n(s^n) g(s^n) \leq \alpha,$$

for some $\alpha > 0$, then, for every sequence s^n:

$$\frac{1}{n!} \sum_{\pi} g(\pi s^n) \leq \alpha (n + 1)^{|\mathcal{S}|}.$$

Proof idea (cont'd)

Second step: Construction of a correlated code using the RT (cont'd)

- (f, φ) - deterministic code for the compound WZ source
- Applying the RT
  Define the codes collection:

$$f_\pi(u^n, s^n) = f(\pi u^n, \pi s^n)$$
$$\varphi_\pi(i, v^n) = \pi^{-1}\varphi(i, \pi v^n).$$

  The RT state that:

$$\frac{1}{n!}\sum_\pi Q(\Delta^\pi(s^n)|s^n) \leq \alpha(n+1)^{|\mathcal{S}|}$$

- 'Random-permutation' code has an exponentially small miss represented set for every state sequence.
- Problem: The number of codes in the collection is exponential with blocklength. $\Longrightarrow$ Can not be transmitted to the decoder without affecting the code rate!

Proof idea (cont'd)

Third step: Elimination technique to rate-distortion

- The randomness in the correlated code should not be exponentially large 'only' polynomial number with blocklength is needed.
- Key idea – LD approach: An empirical average of i.i.d. variables deviates from their mean with exponentially small probability, with the number of variables.
- Select at random only n^2 codes out of the collection. The average miss represented set satisfies:

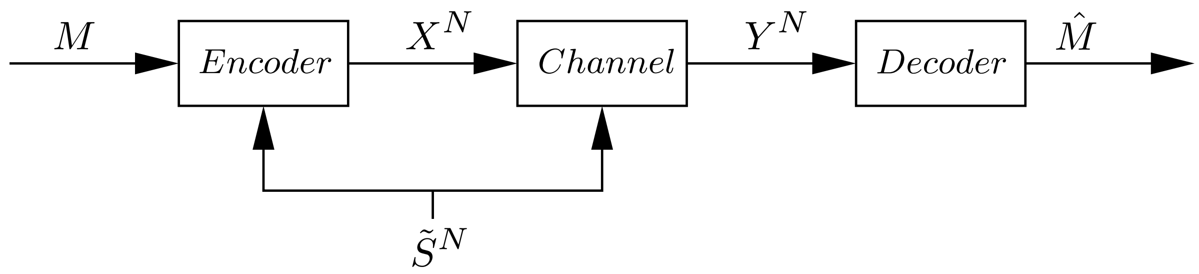
$$P\left\{\max_{s^n} \frac{1}{n^2} \sum_{l=1}^{n^2} P(\Delta^l(s^n)) \geq \lambda\right\} \leq |\mathcal{S}|^n e^{-\lambda n^2}$$

Proof idea (cont'd)

Third step: Elimination technique to rate-distortion (cont'd)

- Random selection argument: there exist a correlated code with only n^2 codes, with a fixed miss represented set probability.
- Create a randomized-encoder code by adding a code index preamble.
- A randomized-encoder code can not out perform a deterministic code. A deterministic code achieves the rate-distortion.
- The miss represented set probability is arbitrary small for sufficiently large blocklength, rather than exponential with blocklength.

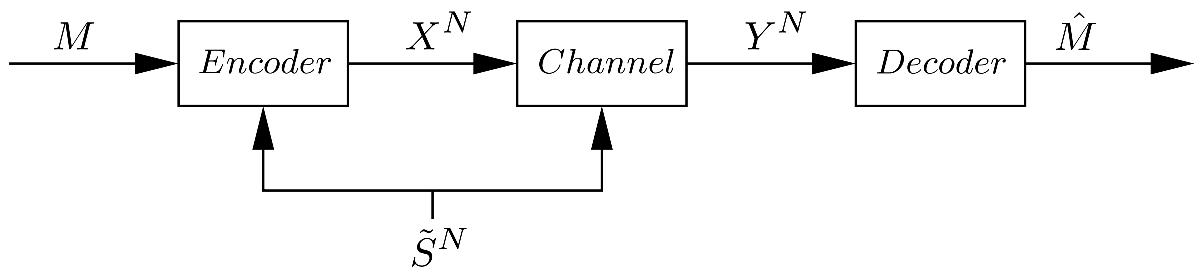
Joint source-channel coding formulation



- $w(y|x, \tilde{s})$ – Memoryless Gel'fand-Pinsker channel with state variable $\tilde{s}$, (AV-GP).
- $\tilde{s^n}$ – State variables, arbitrarily varying, known non-causally at the encoder (CSIT).

We denote the (deterministic code) capacity by C.

Joint source-channel coding formulation (cont'd)



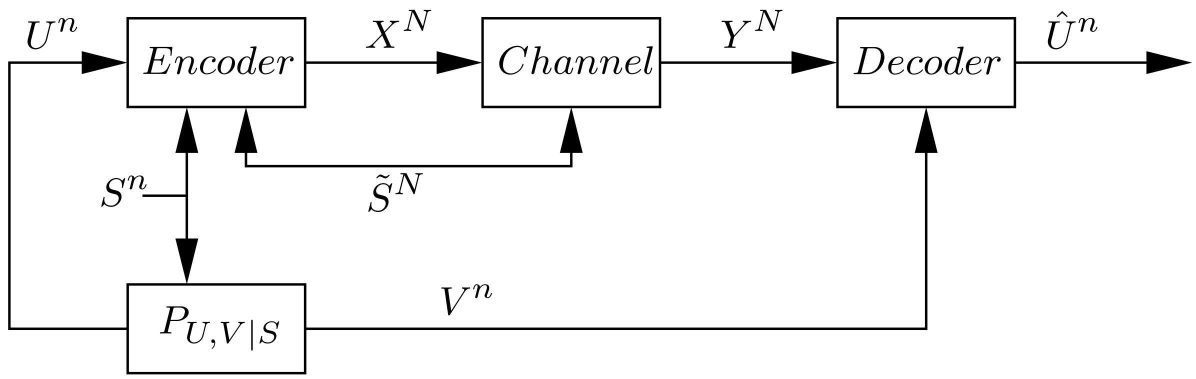
A results by Ahlswede, 1986:

$$C = \min_q C^q$$

where C^q is the capacity of the classical memoryless Gel'fand-Pinsker channel with state distribution q:

$$C^q = \max_{P_{U,X\,|\,S}} [I(U;Y) - I(U;S)].$$

Joint source-channel coding formulation (cont'd)



- An AV-WZ source is transmitted over an AV-GP channel.
- s^n and $\tilde{s}^N$ known non-causally at the encoder.
- V^n - Side information at the decoder.

We are interested in the minimal achievable distortion, for given

$$\rho = \frac{n}{N} = \frac{\rho_s}{\rho_c}.$$

Previous work

- Merhav & Shamai ,2003

  $Q(U, V|S)$ - WZ source with random parameters operate at a rate ρ_s

  $w(y|x, \tilde{s})$ - GP channel with random parameters operate at a rate ρ_c

  A distortion level D is achievable iff:

  $$\rho_s R(D) \leq \rho_c C(\Gamma).$$

  No inherent loss by separating source coding from channel coding.

Joint AV-WZ source and AV-GP channel

- $Q(u, v|s)$ – AV-WZ source, s^n is known at the encoder.
- $R(D)$ – The source rate-distortion function.
- The source operates at a rate of ρ_s symbols per second.
- $w(y|x, s)$ – AV-GP channel with (deterministic code) capacity C.
- The channel operates at a rate of ρ_c channel uses per second.

Theorem:

A distortion level D is achievable iff:

$$\rho_s R(D) \leq \rho_c C.$$

Proof idea

Sufficient part

Shown by concatenating an optimal rate-distortion coding to an optimal channel coding.

Necessary part

If an achievable distortion level D satisfies $\rho_s R(D) > \rho_c C$, then there exists q and q' such that $\rho_s R^q(D) > \rho_c C^{q'}$.

Averaging the distortion with respect to q and q' , results in a contradiction to Merhav-Shamai theorem.

Some observations:

- The best strategy *for the jammer* is to choose the sequences s^n and $\tilde{s}^N$ in such a manner that the Wyner-Ziv source and the Gel'fand-Pinsker channel look like independent of each other.
- A separation principle applies also to the operation of the jammer: the "best" jammer can be split into two non-cooperating jammers, one of which controls only the source state, and the other controls only the channel state, and none of them sees the state sequence of the other.