Coding and Common Knowledge

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Introduction
Motivation

Classical Rate-Distortion Theory:

- The code designer is concerned with reducing the rate $R$ under a constraint on the distortion.
- The question of whether the sender knows $\hat{X}^n$ is not raised.
- $\hat{X}^n = g(f(X^n))$, $\mathbb{E}d(X^n, \hat{X}^n) \leq D$
- $\hat{X}^n$ is known at the encoder.
**Motivation (cont’d)**

The Wyner-Ziv Problem:

![Diagram](image)

- The realization of $Y^n$ is unknown to the encoder.
- $\hat{X}^n = g(f(X^n), Y^n)$, $\mathbb{E}d(X^n, \hat{X}^n) \leq D$.
- The reconstruction $\hat{X}^n$ cannot be reproduced at the sender side.
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In some applications, this is a drawback.
Motivation (cont’d)

Consider the transmission of medical information over noisy BC

- Medical data is sent to a number of experts, for consultation.
  Each has side information $Y_j^n$, $j = 1, 2, \ldots N$ about the patient.
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- The coding scheme guarantees *average* distortion.
  - The distortion pattern is not known at the sender side.
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- The coding scheme guarantees average distortion.
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  Important details can be blurred during transmission. Sender is unaware.
Motivation (cont’d)

Devise a coding scheme that enables the sender to produce locally $\hat{X}_1^n, \ldots, \hat{X}_N^n$. 
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Role of the common reproduction:

- Re-transmit in case that the distortion pattern is “bad.”
- Common reference for the consultation, where the sender and experts know what the data at the other side looks like.
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- Common reference for the consultation, where the sender and experts know what the data at the other side looks like.

$\implies$ Common Knowledge (CK) constraint
Outline

- Source coding with side information at the decoder
- Examples
- Joint source-channel coding for the broadcast channel
- Does common knowledge constraint imply the optimality of a separation-based scheme?
Source coding with side information
Problem formulation

\[ X^n \rightarrow \text{Encoder} \rightarrow T \rightarrow \text{Decoder} \rightarrow \hat{X}^n, \quad \mathbb{E}d(X^n, \hat{X}^n) \leq D \]

rate = \( R \)
Problem formulation (cont’d)

\[ X^n \xrightarrow{\psi} \psi(X^n) \]

Encoder \hspace{1cm} \text{rate} = R \hspace{1cm} \text{Decoder}

\[ Y^n \hspace{1cm} \hat{X}^n, \quad Ed(X^n, \hat{X}^n) \leq D \]
**Problem formulation (cont’d)**

**Definition:** Let $\mathcal{T} = \{1, 2, \ldots, 2^{nR}\}$. An $(n, 2^{nR}, D, \epsilon)$ common knowledge (CK) code for the source $X$ with decoder side information $Y$ consists of an encoder-decoder pair

\[
f : \mathcal{X}^n \to \mathcal{T}, \quad g : \mathcal{T} \times \mathcal{Y}^n \to \hat{X}^n,
\]

and a sender reconstruction map

\[
\psi : \mathcal{X}^n \to \hat{X}^n,
\]

such that

\[
\mathbb{E}d(X^n, g(f(X^n), Y^n)) \leq D,
\]

\[
P_{XY} (\psi(X^n) \neq g(f(X^n), Y^n)) \leq \epsilon.
\]
Problem formulation (cont’d)

$$
\begin{align*}
X^n & \xrightarrow{\psi} \psi(X^n) \\
\text{Encoder} & \quad T \\
\text{Decoder} & \quad Y^n \\
\hat{X}^n, \quad Ed(X^n, \hat{X}^n) & \leq D \\
\end{align*}
$$

$$
\hat{X}^n = g(f(X^n), Y^n)
$$

$$
P_{XY}(\psi(X^n) \neq \hat{X}^n) \leq \epsilon \quad (\text{CK constraint}).
$$
The CK rate-distortion function, $R_{ck}(D)$, is the minimal achievable CK coding rate, under average distortion $D$ and arbitrarily small $\epsilon$. 

$$\hat{X}^n = g(f(X^n), Y^n)$$ 

$$P_{XY}(\psi(X^n) \neq \hat{X}^n) \leq \epsilon \quad \text{(CK constraint)}.$$
Main result

Theorem 1

\[ R_{ck}(D) = \min \{ I(\hat{X}; X) - I(\hat{X}; Y) \} \]

where the minimum is over all \( \hat{X} \) such that \( \hat{X} \leftrightarrow X \leftrightarrow Y \) and

\[ \mathbb{E}d(X, \hat{X}) \leq D. \]
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where the minimum is over all \( \hat{X} \) such that \( \hat{X} \not\subset Y \) and \( \mathbb{E}d(X, \hat{X}) \leq D \).

Due to the Markov conditions, \( R_{ck}(D) = \min I(\hat{X}; X|Y) \).
Comparison with the Wyner-Ziv rate-distortion function

Let $U$ and $\hat{X}$ satisfy $U \circlearrowleft X \circlearrowleft Y$, $\hat{X} \circlearrowleft X \circlearrowleft Y$. Then

\[
R_{WZ}(D) = \min [I(U; X) - I(U; Y)] \quad \mathbb{E}d(X, \phi(U, Y)) \leq D
\]

\[
R_{ck}(D) = \min [I(\hat{X}; X) - I(\hat{X}; Y)] \quad \mathbb{E}d(X, \hat{X}) \leq D
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- In the WZ problem, the codewords $U^n$ need not satisfy the distortion constraint by themselves. The side information $Y$ is used for binning and estimation.
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- There is no external random variable in the CK problem.
Typical curves

\[ R_{ck}(D) = \min[I(\hat{X}; X) - I(\hat{X}; Y)] \quad \mathbb{E}d(X, \hat{X}) \leq D \]

\[ X \ominus X \ominus Y \]
Examples

Example 1  *The doubly symmetric binary source, Hamming distortion measure*

\[ P_X(0) = \frac{1}{2} \]

\[ P_X(1) = \frac{1}{2} \]

\[ Y = X \oplus Z, \quad Z \sim \text{Bernoulli}(p_z) \]

\[ R_{ck}(D) = h(p_z \ast D) - h(D), \quad 0 \leq D \leq \frac{1}{2}. \]
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  R_{ck}(D) &= h(p_z \ast D) - h(D), \quad 0 \leq D \leq 1/2.
\end{align*}
\]

Closely related to the Wyner-Ziv rate-distortion function

\[
R_{WZ}(D) = \text{l.c.e.} \{h(p_z \ast D) - h(D), (p_z, 0)\}.
\]
Examples – doubly symmetric source (cont’d)

Corollary 1 For the binary doubly symmetric source with Hamming distortion measure, no penalty is incurred due to the common knowledge constraint in the region $0 \leq D \leq D_c$. 

![Diagram showing R(D), Rck(D), and RWZ(D) curves]
Examples (cont'd)

**Example 2**  *Gaussian source and square error distortion measure.*

\[ Y = X + Z, \quad X \sim \mathcal{N}(0, \sigma_X^2), \quad Z \sim \mathcal{N}(0, \sigma_Z^2), \quad X \perp V \]
**Examples (cont'd)**

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\[
R_{ck}(D) = \frac{1}{2} \log \left( \frac{\sigma_X^2 \sigma_Z^2}{(\sigma_X^2 + \sigma_Z^2)D} \cdot \frac{D + \sigma_Z^2}{\sigma_Z^2} \right).
\]
Examples (cont’d)

Example 2  *Gaussian source and square error distortion measure.*

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\]

Note that

\[
R_{WZ}(D) = R_{X|Y}(D) = \frac{1}{2} \log \left( \frac{\sigma^2_X \sigma^2_Z}{(\sigma^2_X + \sigma^2_Z)D} \right),
\]

therefore \( \frac{1}{2} \log \left( \frac{D + \sigma^2_Z}{\sigma^2_Z} \right) \) is the penalty due to the CK constraint.
Examples – Gaussian source (cont’d)

Here $\sigma_X^2 = \sigma_Z^2 = 1$. 

\[ R(D) \quad R_{ck}(D) \quad R_{WZ}(D) \]
Joint source-channel coding for the BC
Problem formulation

\[ X^n \xrightarrow{U^m} V_1^m \xrightarrow{V_2^m} \hat{X}_1^n, \quad \mathbb{E}d(X^n, \hat{X}_1^n) \leq D_1 \]

\[ X^n \xrightarrow{U^m} V_2^m \xrightarrow{V_1^m} \hat{X}_2^n, \quad \mathbb{E}d(X^n, \hat{X}_2^n) \leq D_2 \]
Problem formulation

\[
X^n \xrightarrow{\psi_1} \psi_1(X^n) \quad \psi_2(X^n) \xrightarrow{\psi_2} \psi_2(X^n)
\]

Encoder \quad U^m \xrightarrow{V_1^m} \xrightarrow{V_2^m} \text{Channel} \quad \text{Decoder 1} \quad \hat{X}_1^n, \ I_{Ed}(X^n, \hat{X}_1^n) \leq D_1

Decoder 2 \quad \hat{X}_2^n, \ I_{Ed}(X^n, \hat{X}_2^n) \leq D_2
Problem formulation

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- Bandwidth expansion ratio $\rho = m/n$. 

Diagram:

- $X^n$ is encoded by the encoder.
- The encoded signal $U^m$ is transmitted through the channel.
- The channel outputs $V_1^m$ and $V_2^m$.
- Decoder 1 receives $V_1^m$ and decodes $\hat{X}_1^n$, satisfying $\mathbb{E}d(X^n, \hat{X}_1^n) \leq D_1$.
- Decoder 2 receives $V_2^m$ and decodes $\hat{X}_2^n$, satisfying $\mathbb{E}d(X^n, \hat{X}_2^n) \leq D_2$.
- $\psi_1$ and $\psi_2$ are transformations applied to the encoded signal.
Problem formulation

- Memoryless, degraded broadcast channel $U \rightarrow V_2 \rightarrow V_1$.
- Bandwidth expansion ratio $\rho = m/n$.
- CK constraints:
  \[
P(\psi_j(X^n) \neq \hat{X}_j^n) \leq \epsilon, \quad j = 1, 2.
\]
Main result

- $\mathcal{C}$ – the capacity region of the degraded broadcast channel $P_{V_1,V_2|U}$
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- $\mathcal{R}_X(D_1, D_2)$ – the successive refinement rate region of the source $X$ at distortions $(D_1, D_2)$
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**Theorem 2**  Under the CK constraint, the distortion pair $(D_1, D_2)$ is achievable with bandwidth expansion ratio $\rho$ if and only if

$$\mathcal{R}_X(D_1,D_2) \cap \rho \mathcal{C} \neq \emptyset$$
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$\implies$ Separation yields optimal distortion pairs.
CK and separation
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  - Separation is optimal.

If CK requires the distortion to be introduced by the sender, is separation always optimal under CK?
Does CK imply separation?

Unfortunately, there are situations where separation is suboptimal even under the CK constraint.
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- Separation is suboptimal in \textit{lossless} transmission of a joint source over the MAC (Cover, El Gamal, andSalehi 1980)
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Unfortunately, there are situations where separation is suboptimal even under the CK constraint

- Separation is suboptimal in *lossless* transmission of a joint source over the MAC (Cover, El Gamal, and Salehi 1980)
- Separation is suboptimal in lossy transmission of state over the state dependent channel, even under CK constraint.
Thank You