Problems We Can Solve With a Helper

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Outline

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  - The rate-distortion region, and important observations

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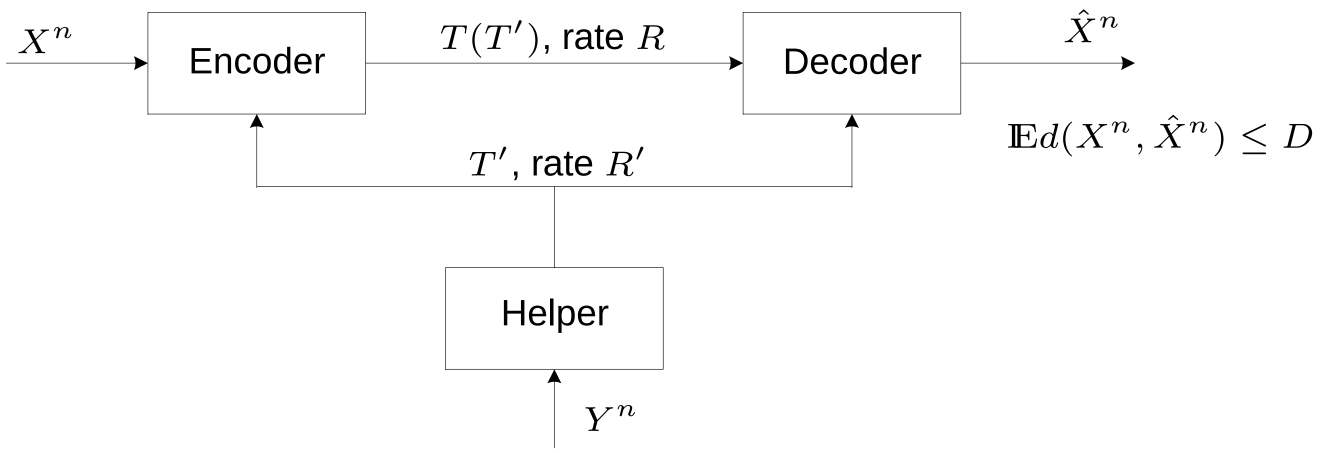
- Cascade rate-distortion with a helper
- Examples

Problem Formulation



Goal: Compression of X^n, with rate-limited SI at both ends:

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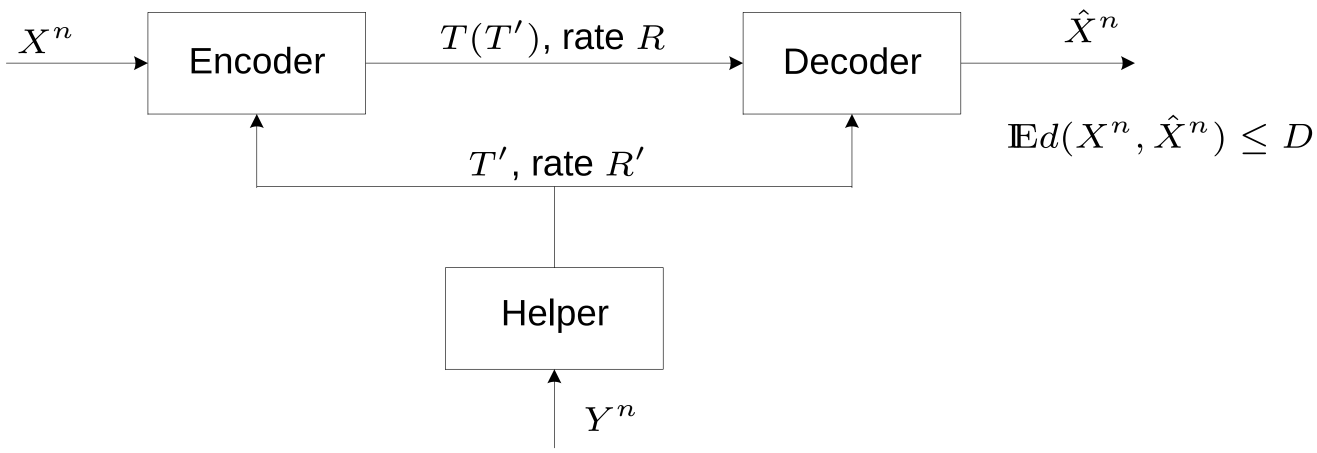
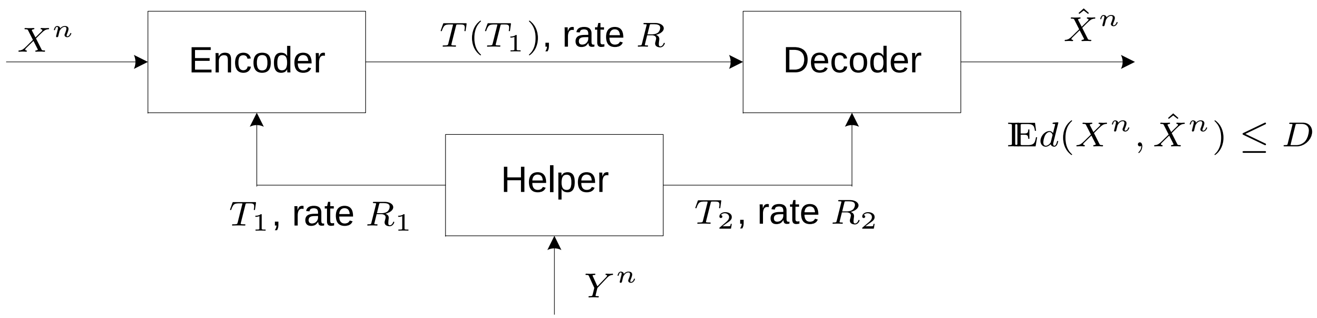


Goal: Compression of X^n, with rate-limited SI at both ends:

A. Common

Problem Formulation



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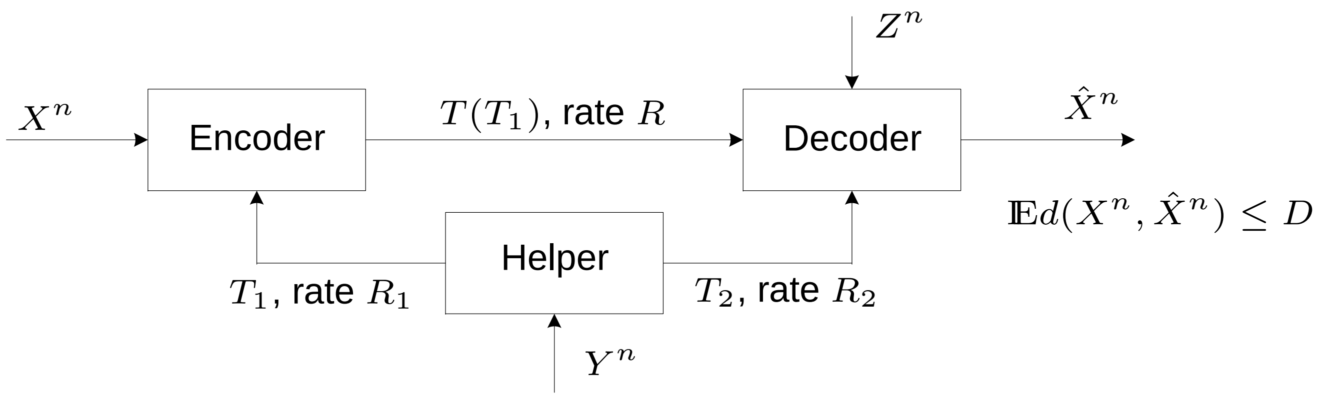

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A. Common

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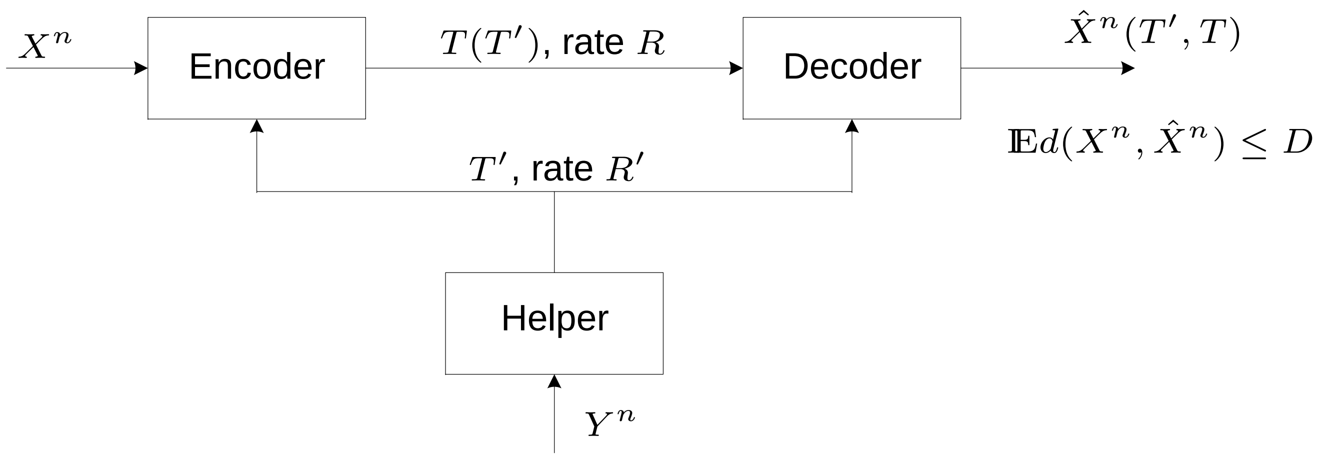
C. With decoder SI

Problem Formulation



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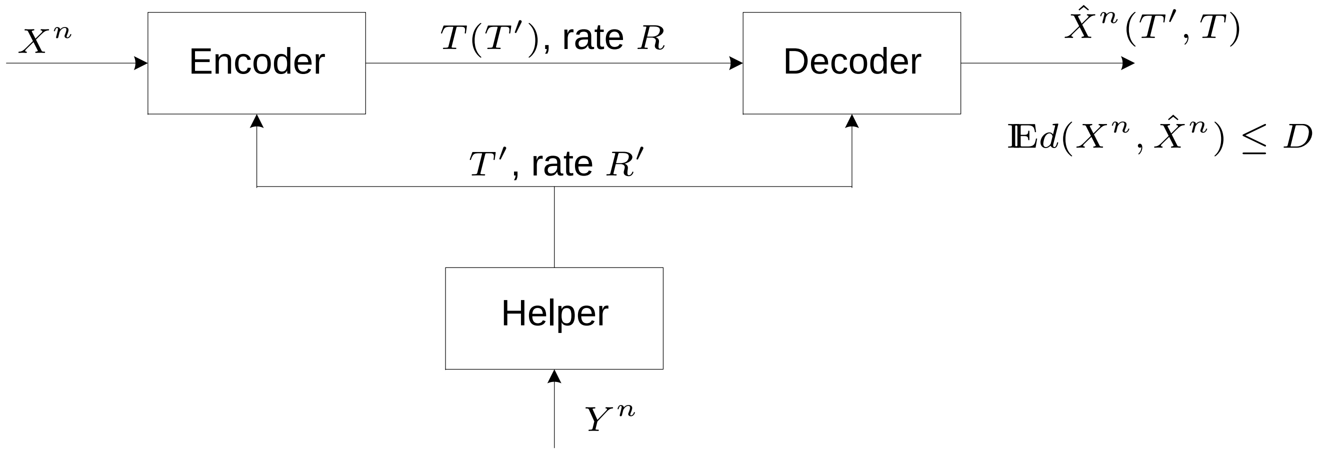

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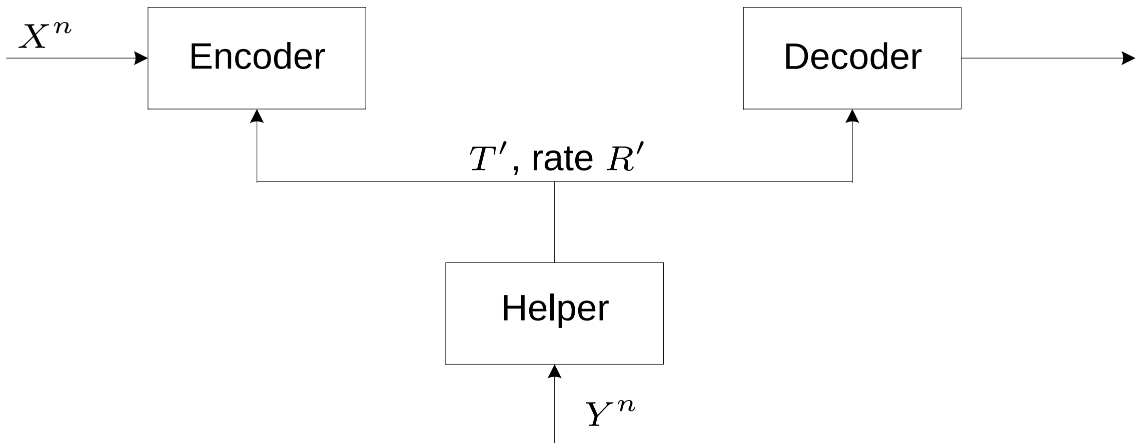

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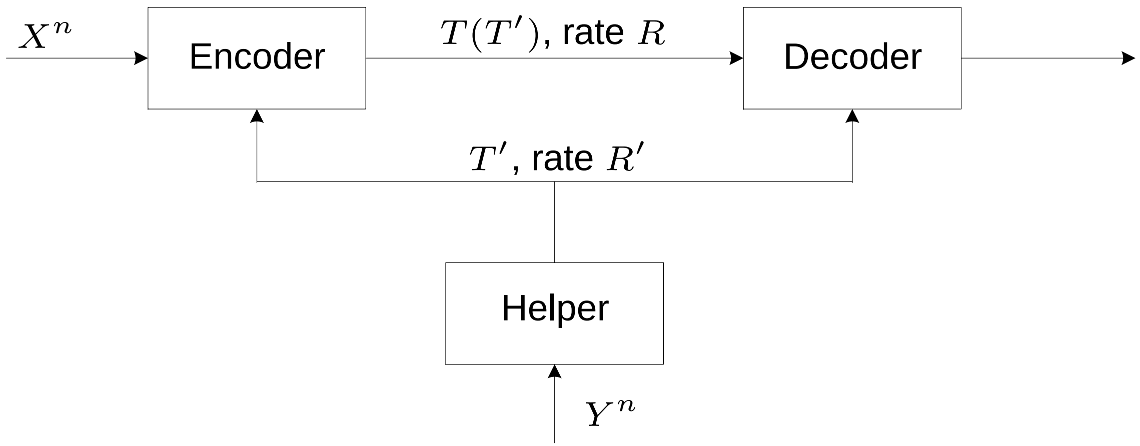

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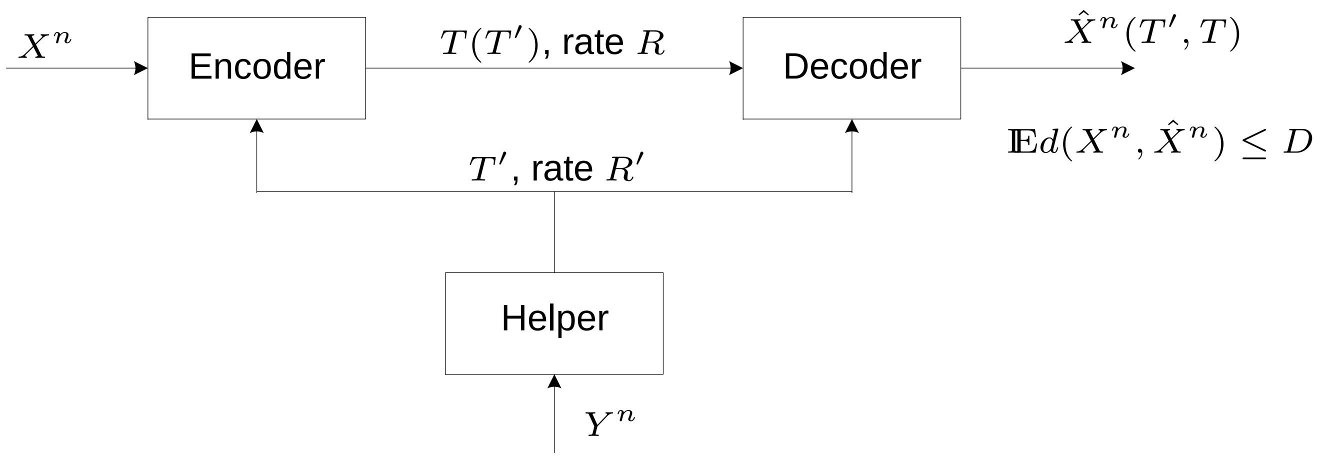

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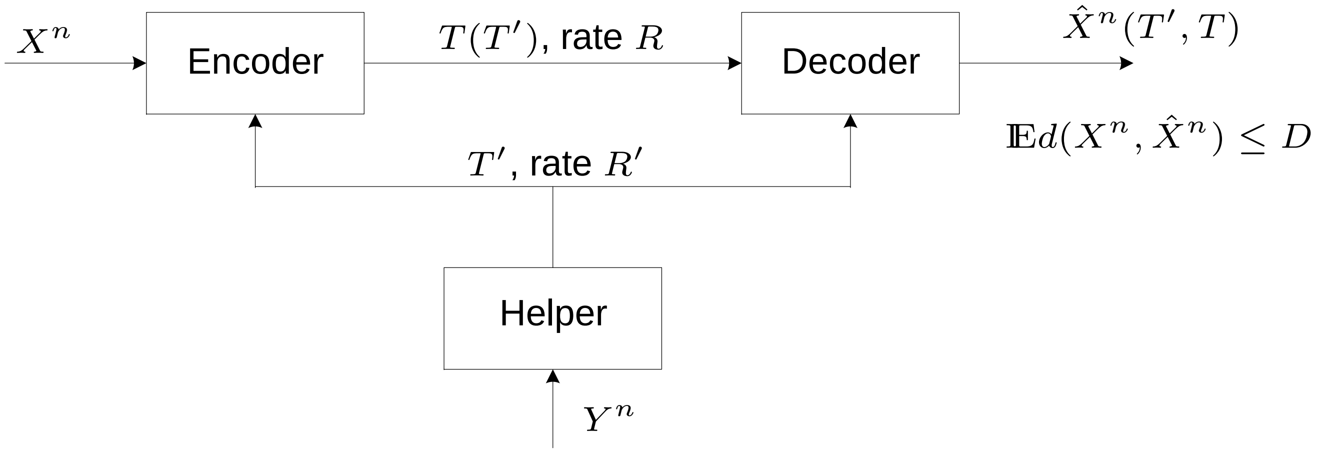
3. The destination constructs an estimate $\hat{X}^n$, based on T and T'.

Problem Formulation *(cont'd)*



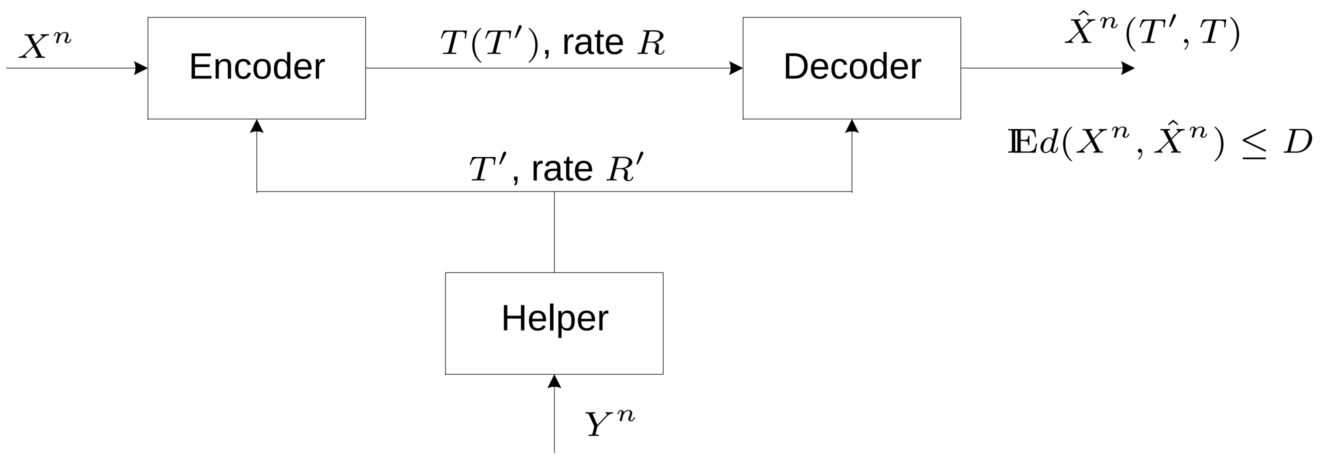
Model:

$(X, Y) \sim P_{X,Y}$, memoryless.

We are interested in the region of all achievable rate pairs and distortion

$$(R, R', D).$$

Problem Formulation *(cont'd)*



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$\mathcal{R}(D)$ – the collection of all rate pairs (R, R') such that the triplet (R, R', D) is achievable.

Previous Work

- Lossless source coding with rate-limited SI at the decoder(s)

  Wyner 75.

  Ahlswede and Körner, 75

- Lossy source coding with SI at the decoder

  Wyner and Ziv, 76.

- Rate-distortion for correlated sources with partially separated encoders

  Kaspi, Ph.D. dissertation, 79.

  Kaspi and Berger, IEEE IT 1982.

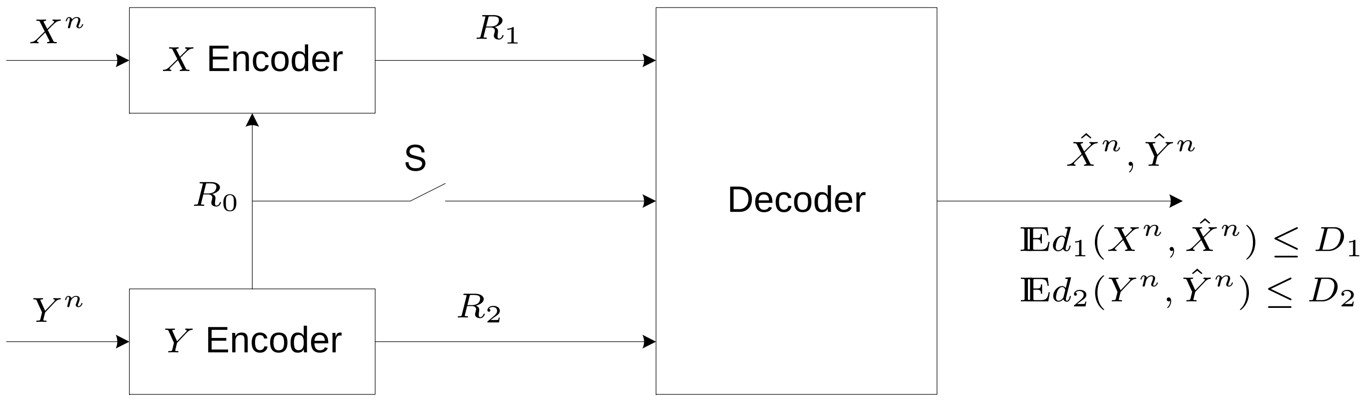
  Subsumes all previous models.

Previous Work *(cont'd)*

- Cooperative source coding with encoder breakdown,
  Dinkar Vasudevan and Etienne Perron, ISIT 2007, Nice.

Previous Work *(cont'd)*

The Kaspi and Berger model:



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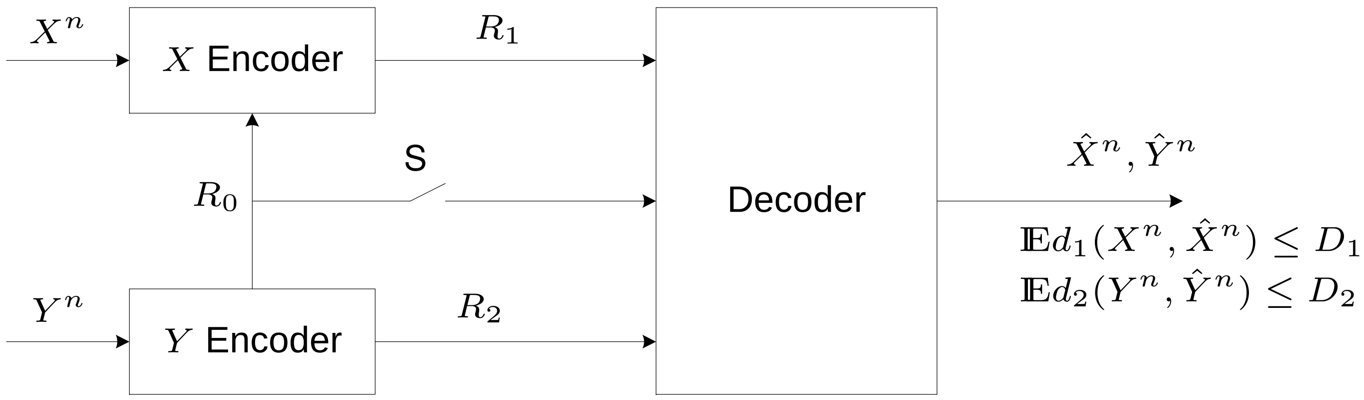

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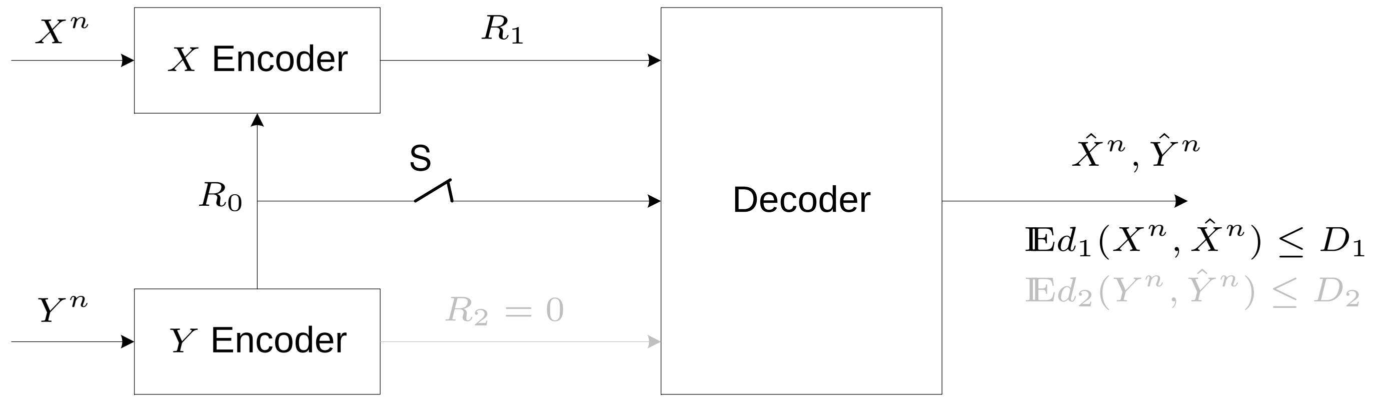

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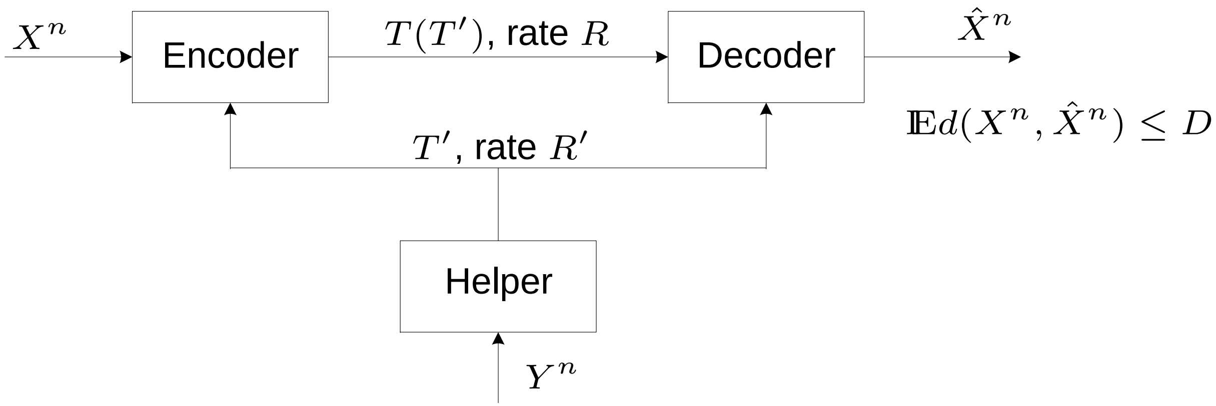
The rate-distortion region with common messages



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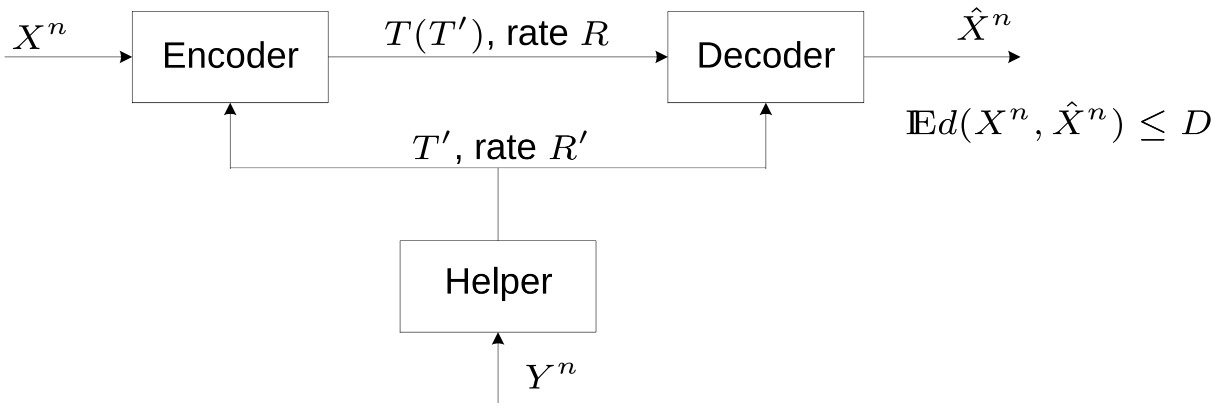


Theorem 1 [Vasudevan & Perron, 2007, PSW 2008]
$(R, R') \in \mathcal{R}(D)$ if and only if

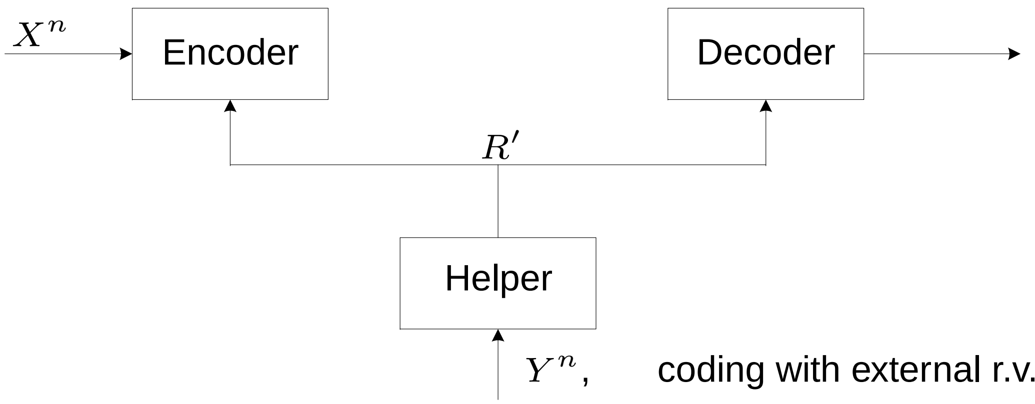
$$\begin{aligned} R' &\geq I(U;Y) \\ R &\geq I(X;\hat{X}|U) \\ D &\geq \mathbb{E}d(X,\hat{X}) \end{aligned}$$

for some joint distribution of the form

$$P_{X,Y} \cdot P_{U|Y} \cdot P_{\hat{X}|U,X}.$$

The R-D region with common messages *(cont'd)*

Direct Part



coding with external r.v. U

The R-D region with common messages (cont'd)

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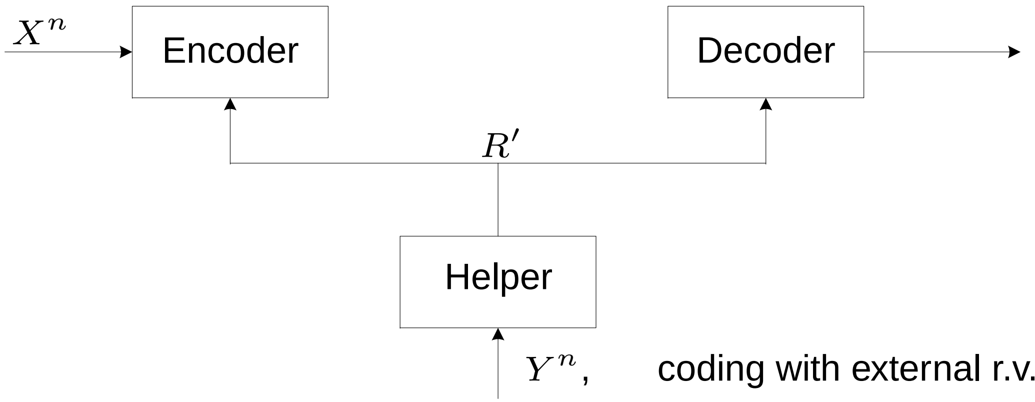

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$$R' \geq I(U;Y) \qquad U - Y - (X, \hat{X})$$

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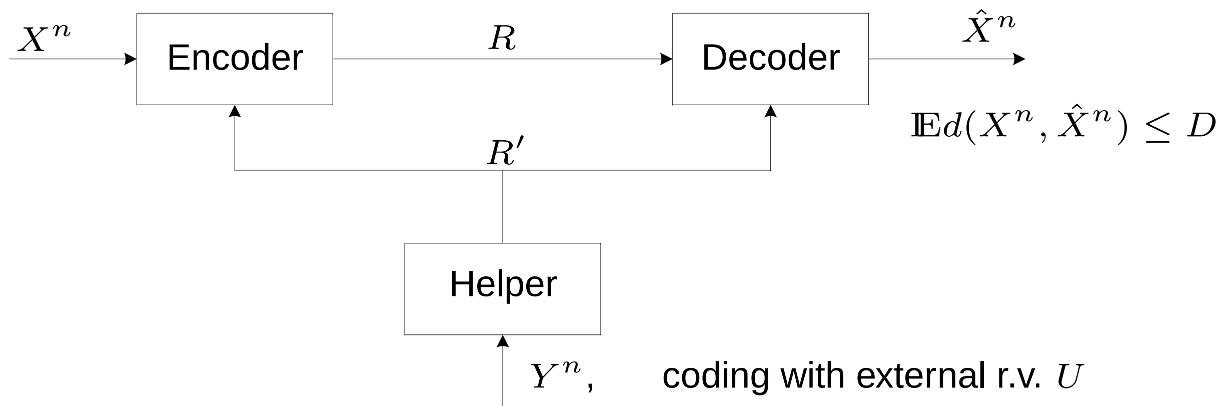

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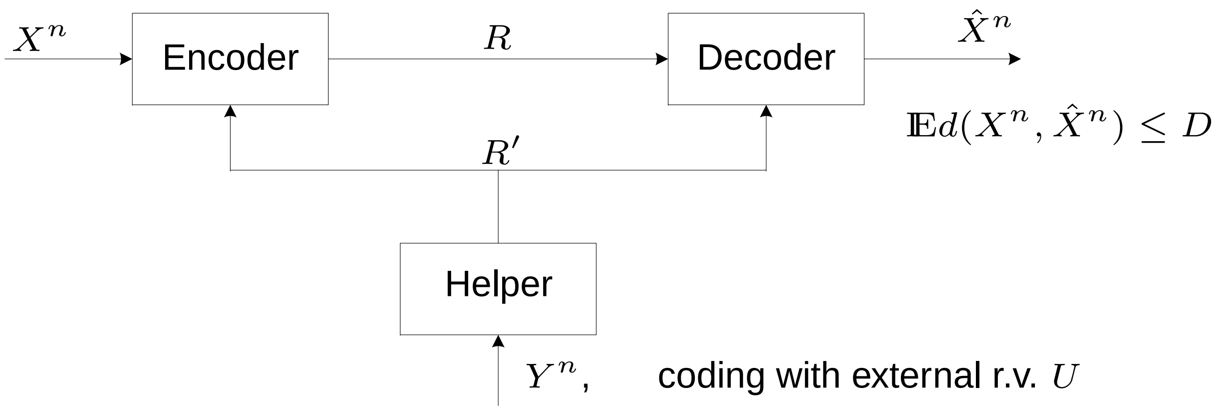

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$$R' \geq I(U;Y) \qquad U - Y - (X, \hat{X})$$

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$$R \geq I(X;\hat{X}|U) \qquad \hat{X} - (U,X) - Y$$

$$D \geq \mathbb{E}d(X,\hat{X})$$

The R-D region with common messages (cont'd)

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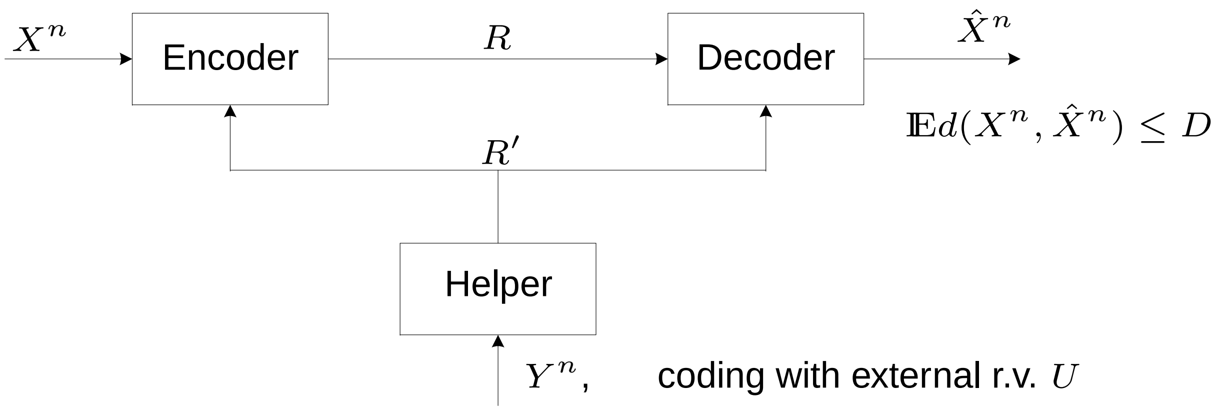

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$$P_{X,Y} \cdot P_{U|Y} \cdot P_{\hat{X}|U,X}.$$

The R-D region with common messages (cont'd)

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In most converse proofs, the main issue is to get the Markov conditions.

Do we really need them?

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$$\Longrightarrow \qquad P_{\hat{X},U,X,Y} \text{ need not be kept. Only } P_{\hat{X},U,X}.$$

The R-D region with common messages *(cont'd)*

Another way of saying the same thing: Let

$$\tilde{P}_{\hat{X},U,X,Y} = P_{X,Y} \cdot P_{U|Y} \cdot P_{\hat{X}|U,X,Y}$$
$$P_{\hat{X},U,X,Y} = P_{X,Y} \cdot P_{U|Y} \cdot P_{\hat{X}|U,X}$$

with $P_{\hat{X}|U,X}$ the conditional marginal induced by $\tilde{P}_{\hat{X},U,X,Y}$.

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The region

$$R' \geq I(U;Y)$$
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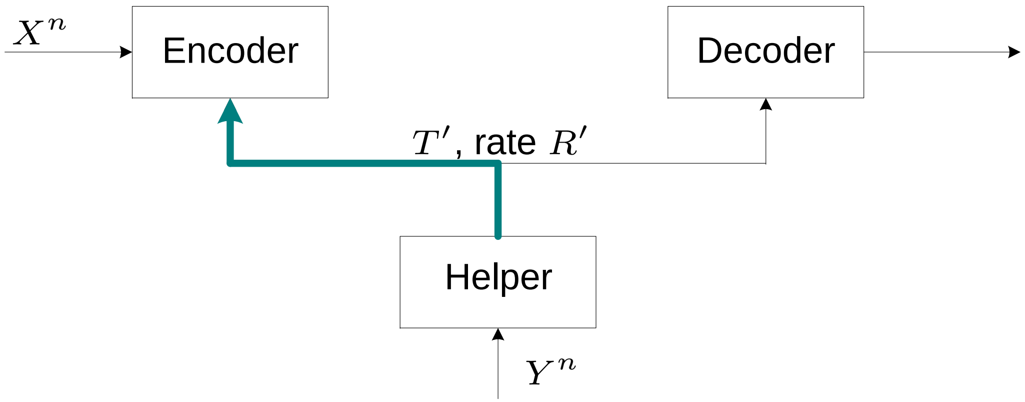
Thus, it is enough to prove:

Achievability for any $P_{X,Y} \cdot P_{U|Y} \cdot P_{\hat{X}|U,X}$

Upper bound for $P_{X,Y} \cdot P_{U|Y} \cdot P_{\hat{X}|U,X,Y}$

A Second Look at the Helper



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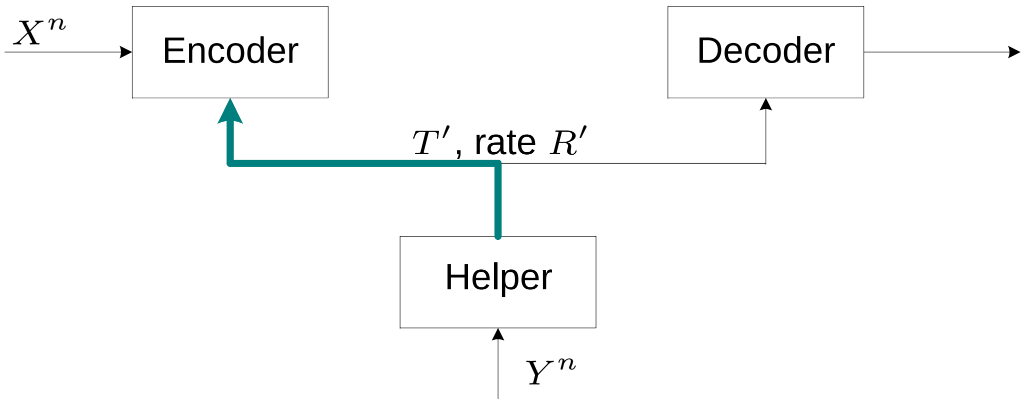

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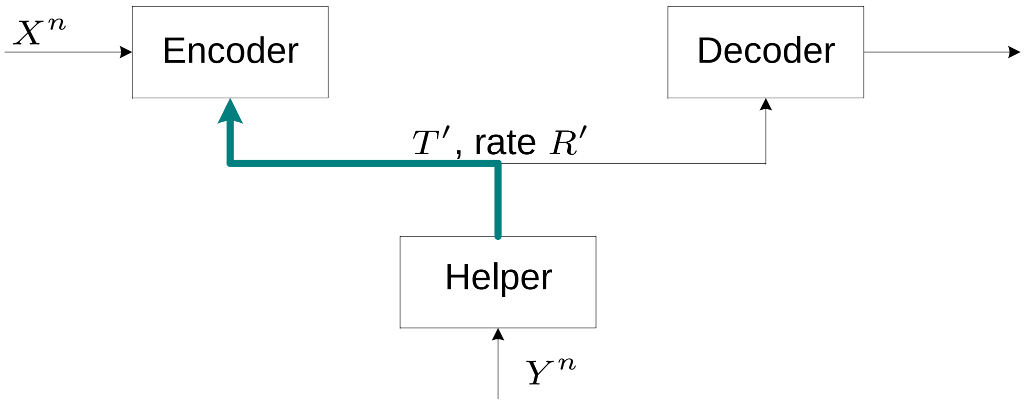

- The same stream T' is sent to the encoder and decoder
- The encoder has X^n at hand, which can serve as SI, to further improve its knowledge about Y^n

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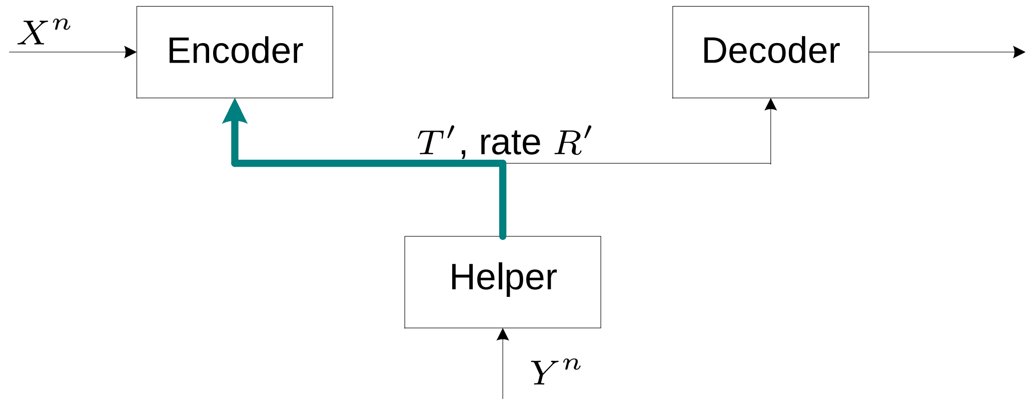

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$\Rightarrow$ “Rate distortion when side information may be absent," Heegard and Berger, *IEEE IT*, Nov. 1985.

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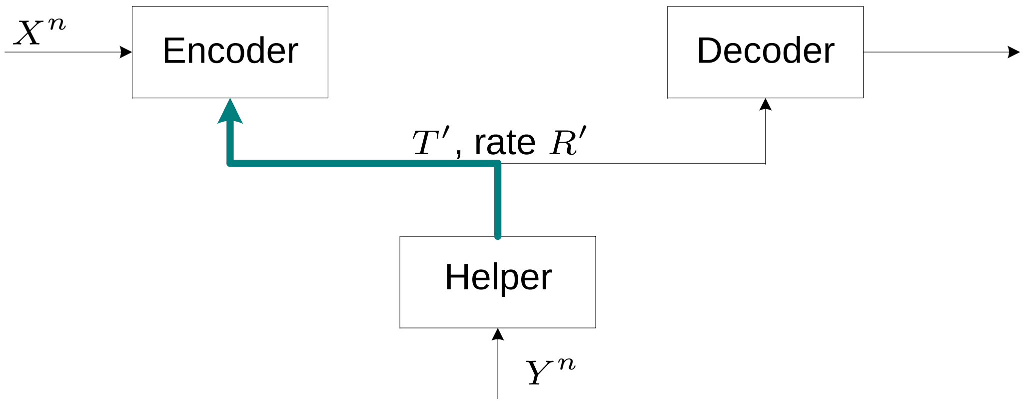

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$$R' \geq I(U;Y) + I(V;Y|U,X), \qquad (U,V) - Y - X.$$

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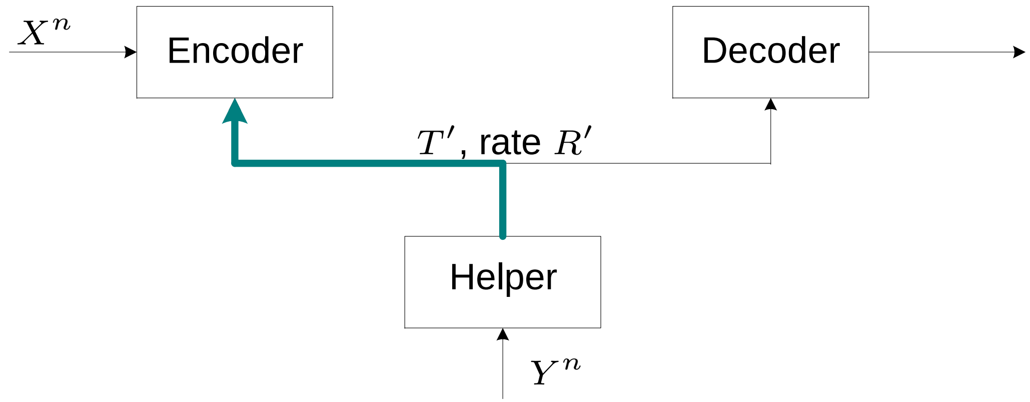

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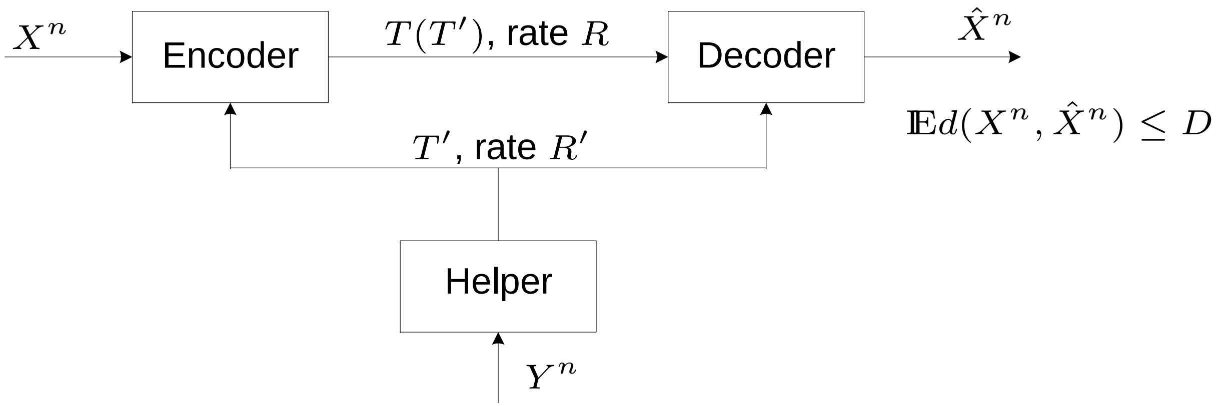
But in our result, we have only $R' \geq I(U;Y)$.

A Second Look at the Helper (cont'd)



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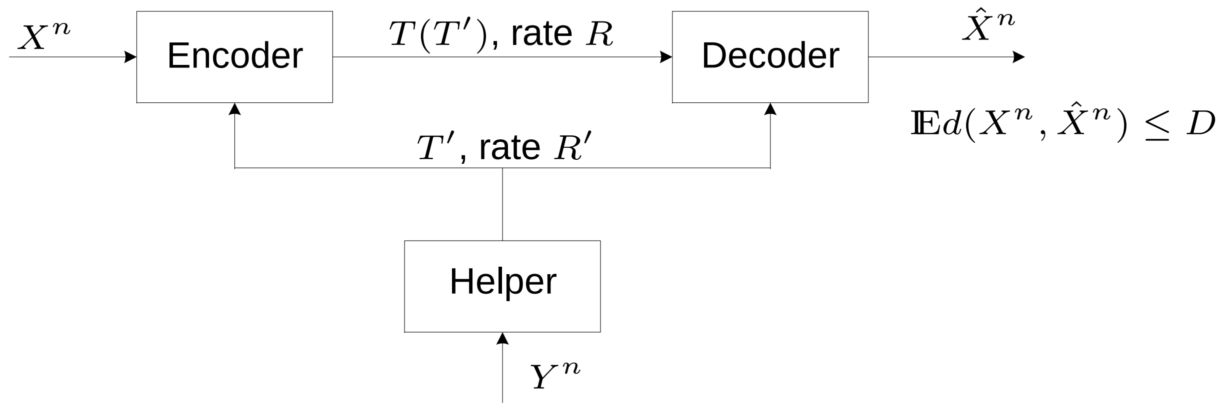

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When sending common message to the encoder and decoder, the source encoder does not use its SI X^n to improve his knowledge on Y^n beyond what is known at the decoder.

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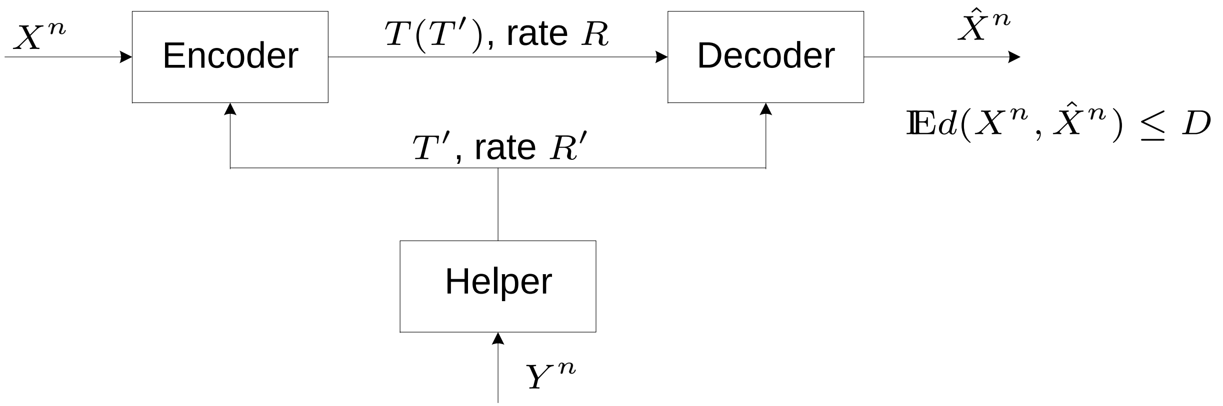

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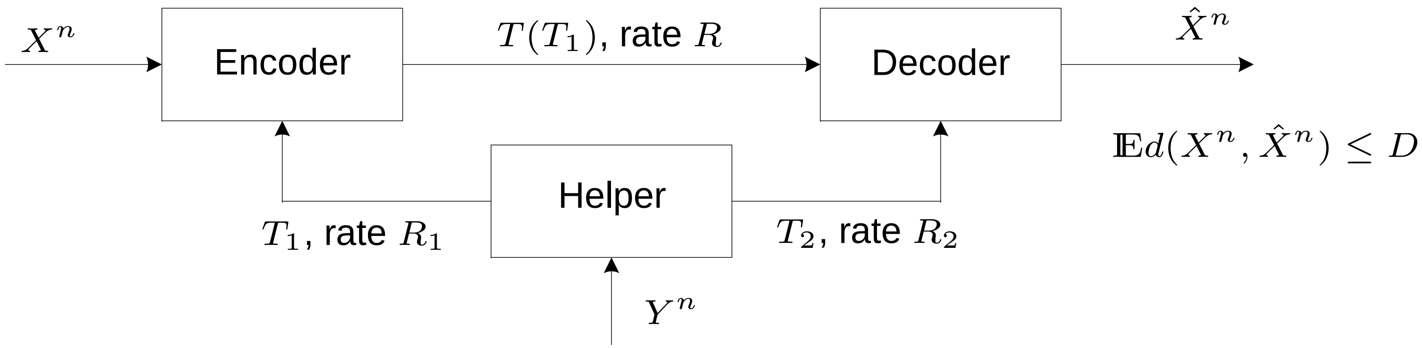
Will he use extra rate?

Independent rates



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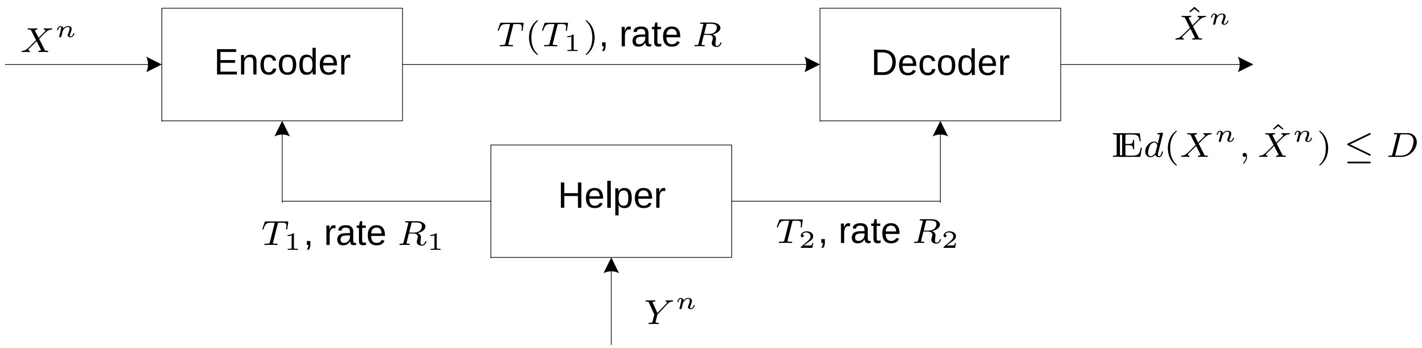

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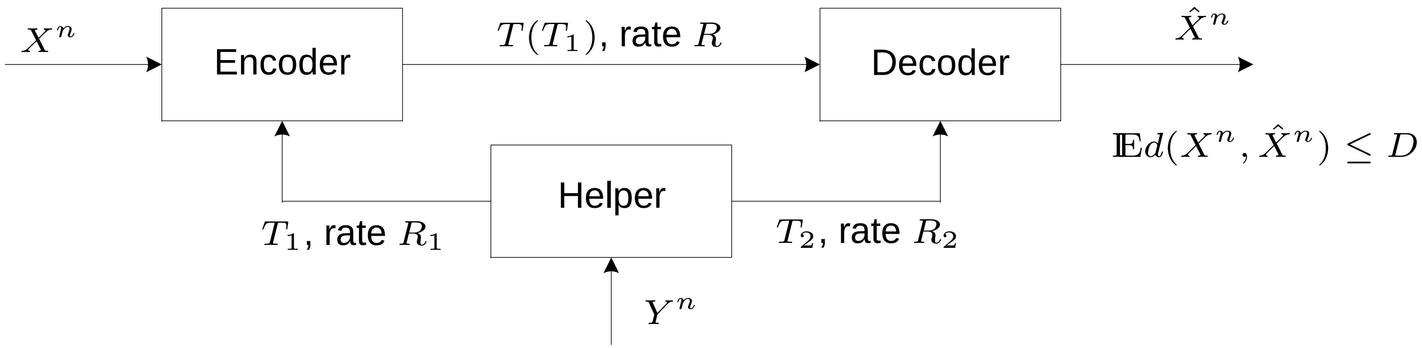

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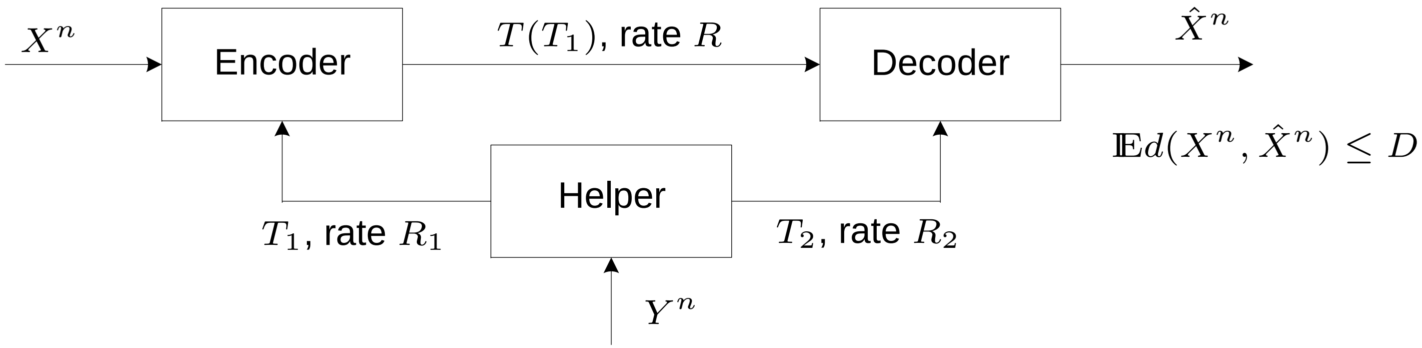

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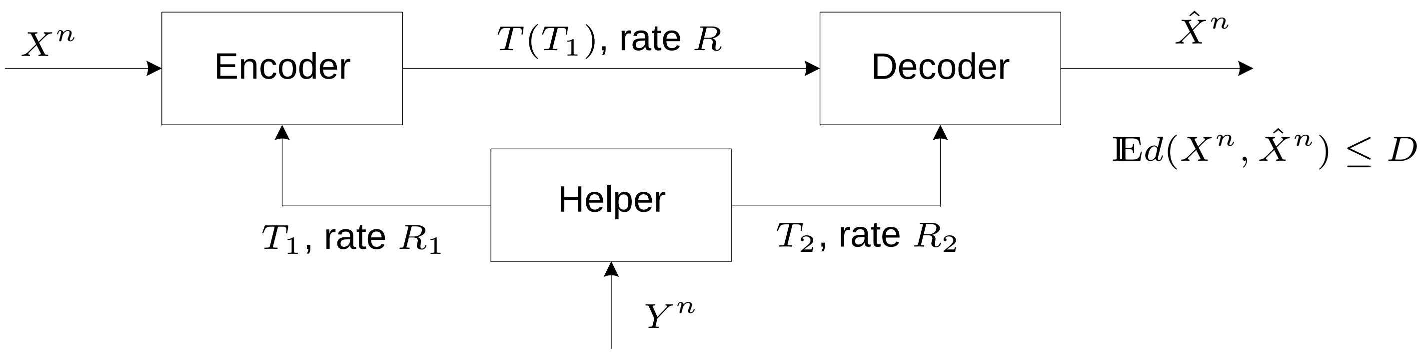
Proof – Operational arguments.

Extra helper rate to the encoder is not used. Moreover, even when independent messages are allowed, sending a common helper message yields optimal performance. ($R_1 \geq R_2$).

Independent rates (cont'd)

What about reducing R_1 below R_2?



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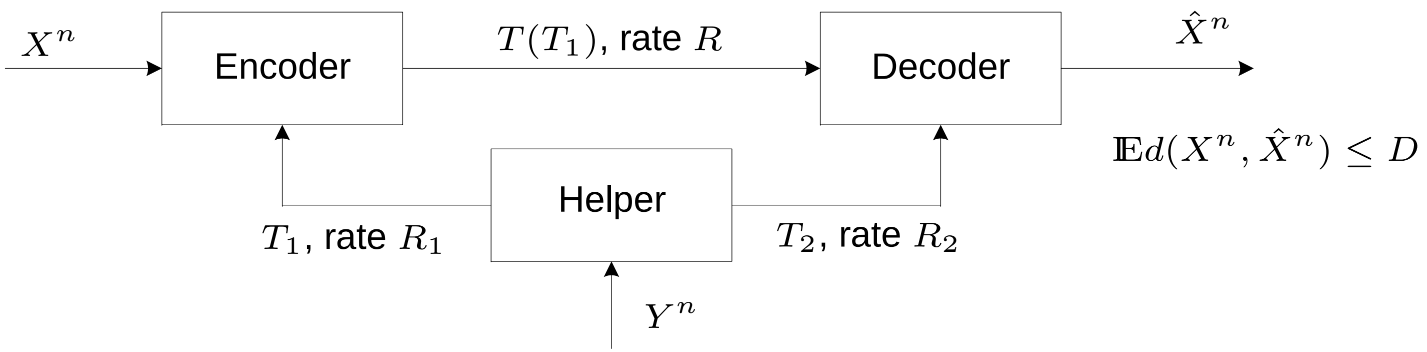

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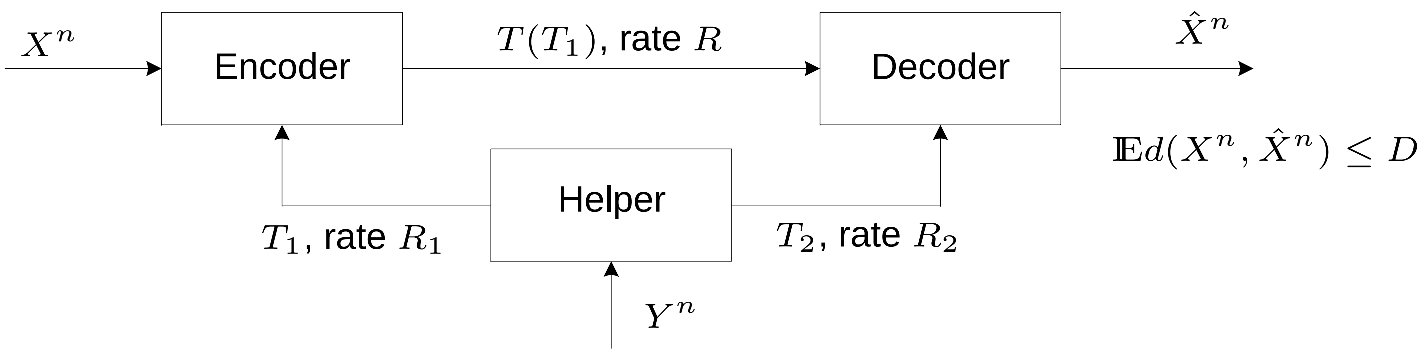

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Independent rates *(cont'd)*

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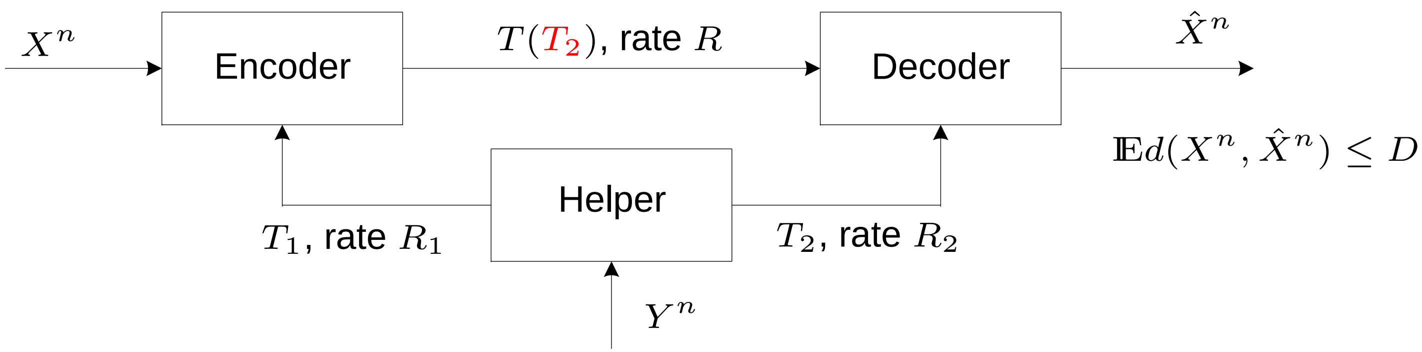

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$$\begin{aligned} R_2 &\ge I(U;Y) \\ R &\ge I(X;\hat{X}|U) \\ D &\ge \mathbb{E}d(X,\hat{X}) \end{aligned}$$

We can reduce R_1 slightly below R_2, and the encoder still be able to decode T_2:

$$R_1 \ge I(U;Y) - I(U;X)$$

Example

Doubly symmetric binary source with Hamming distortion measure

$$X = Y \oplus Z, \quad Y \sim \text{Bernoulli}(\frac{1}{2}), \quad Z \sim \text{Bernoulli}(p_z)$$

The region with common helper:

$$R \geq H_b(H_b^{-1}([1 - R']_+) * p_z) - H_b(D).$$

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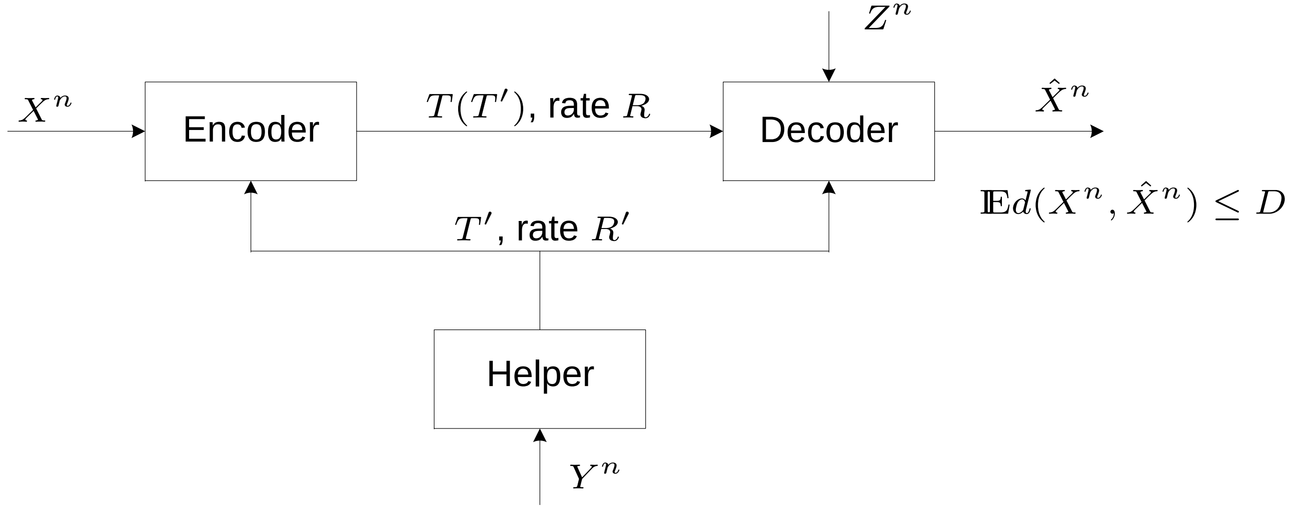
The region with two rates, $R_2 \geq R_1$

$$\begin{aligned} R &\geq H_b(H_b^{-1}([1 - R_2]_+) * p_z) - H_b(D) \\ R_1 &\geq H_b(H_b^{-1}([1 - R_2]_+) * p_z) - [1 - R_2]_+. \end{aligned}$$

Common Helper and Decoder SI



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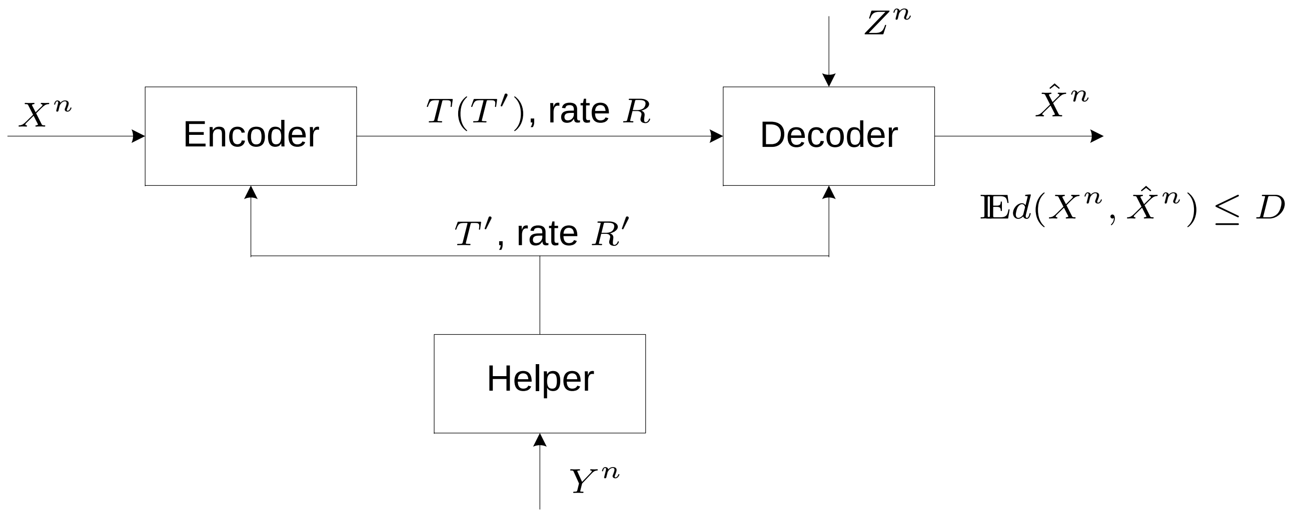
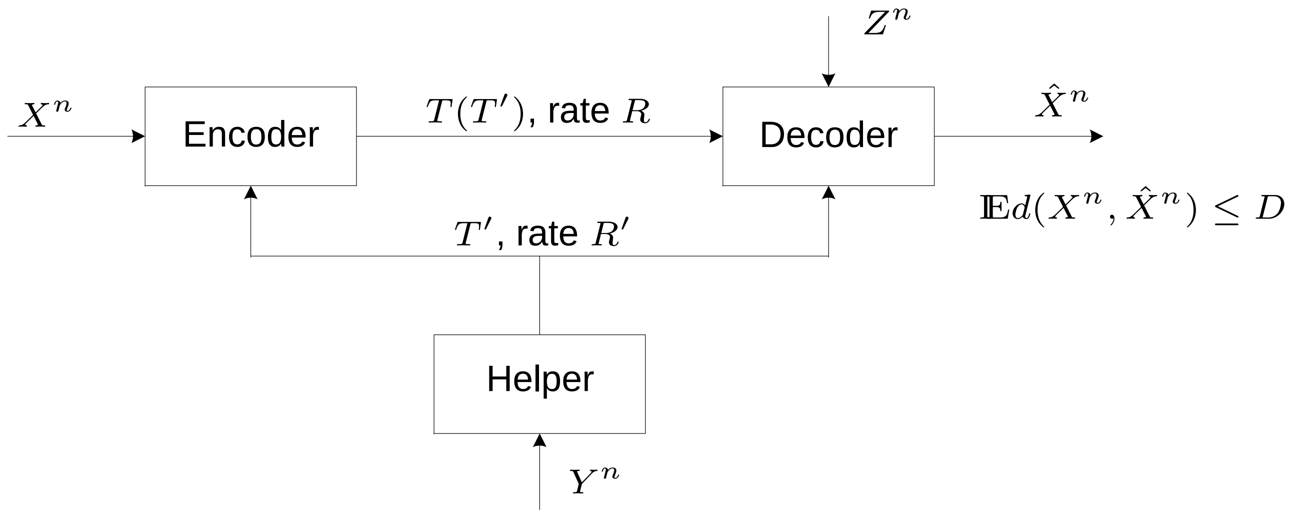


- We assume the Markov chain $Y - X - Z$

Common Helper and Decoder SI



- We assume the Markov chain $Y - X - Z$
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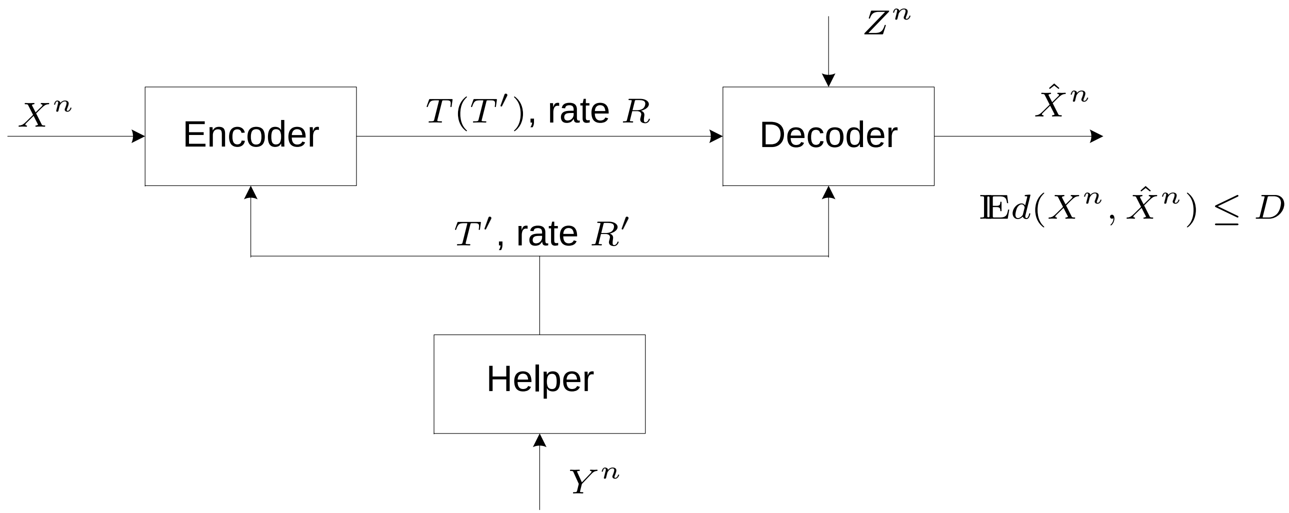
Common Helper and Decoder SI



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- The rate distortion region is denoted $\mathcal{R}(D|Z)$.

Main Result

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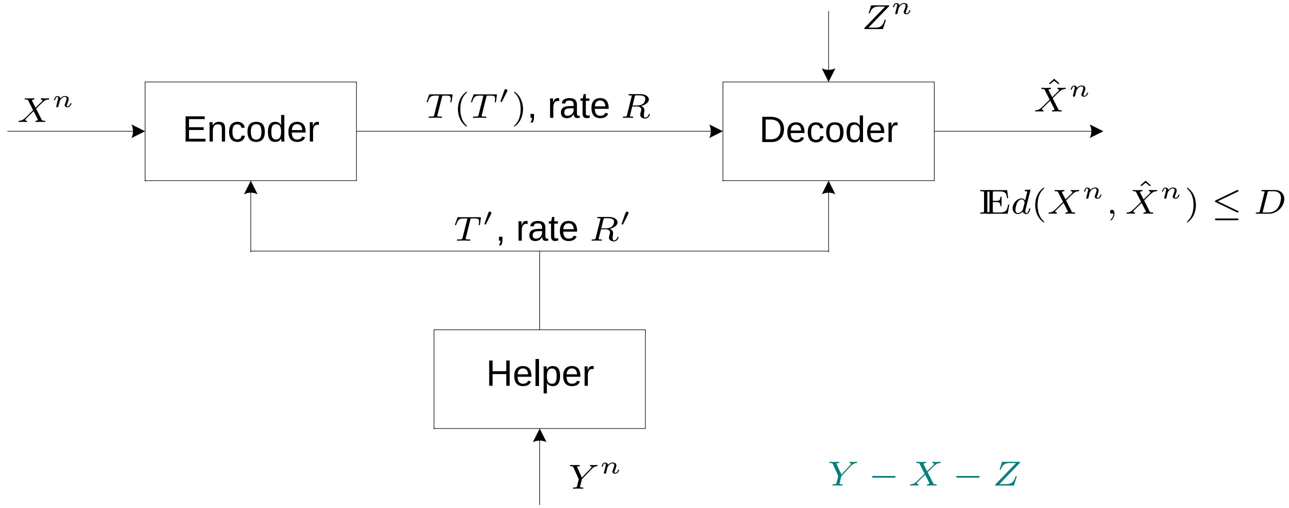
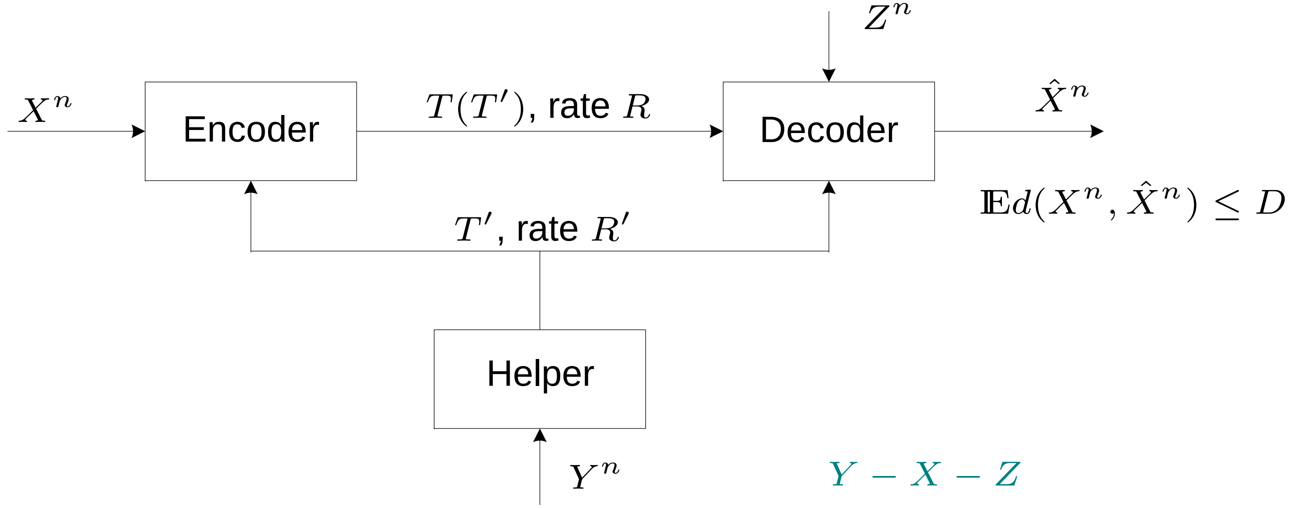


$Y - X - Z$

Main Result



$Y - X - Z$

Theorem 3 $(R, R') \in \mathcal{R}(D|Z)$ if and only if

$$\begin{aligned} R' &\geq I(U;Y|Z) \\ R &\geq I(V;X|U,Z) \\ D &\geq \mathbb{E}d(X, \hat{X}(U,V,Z)) \end{aligned}$$

for some joint distribution of the form

$$P_{X,Y} \cdot P_{Z|X} \cdot P_{U|Y} \cdot P_{V|U,X}.$$

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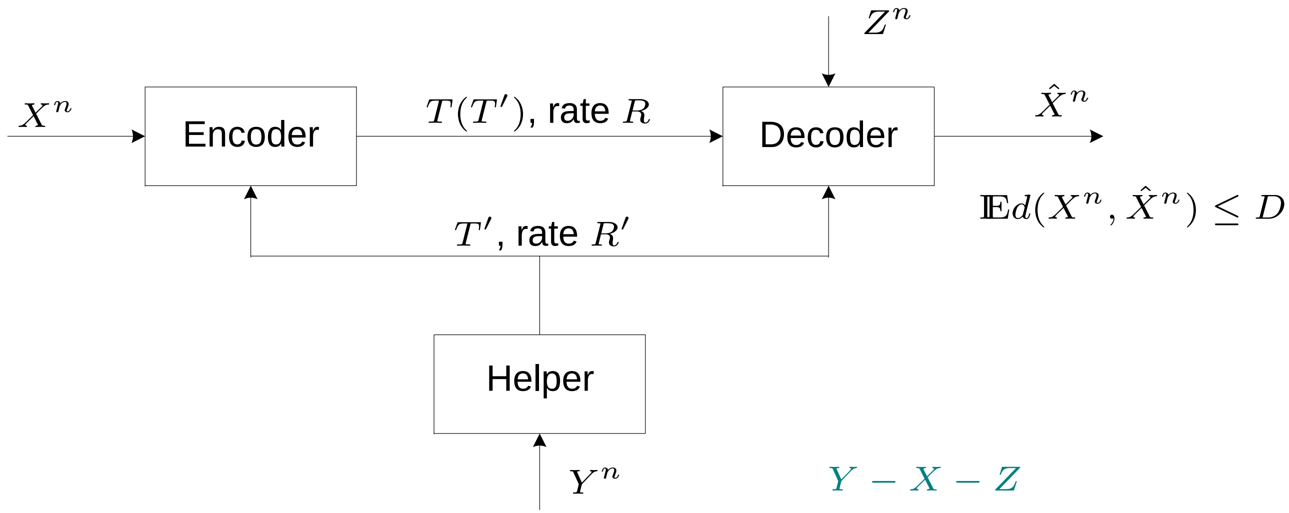

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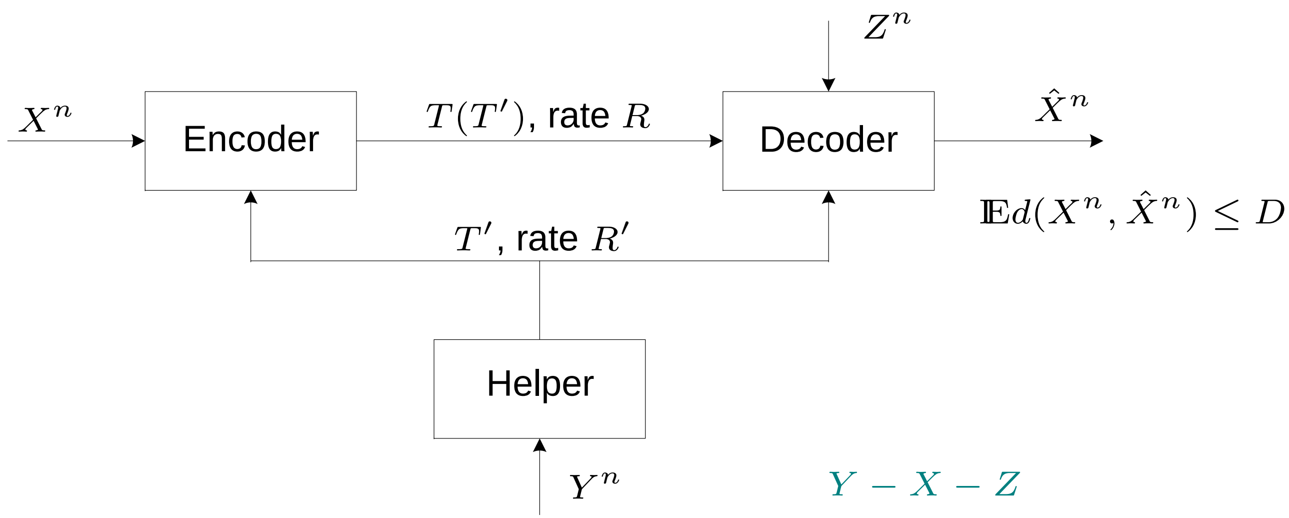

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Markov Invariancy

Let

$$\begin{aligned} \tilde{P}_{U,V,X,Y,Z} &= P_{X,Y} \cdot P_{Z|X} \cdot P_{U|Y} \cdot P_{V|U,X,Y} \\ P_{V,U,X,Y,Z} &= P_{X,Y} \cdot P_{Z|X} \cdot P_{U|Y} \cdot P_{V|U,X} \end{aligned}$$

with $P_{V|U,X}$ the conditional marginal induced by $\tilde{P}_{U,V,X,Y,Z}$.

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The region

$$R' \geq I(U;Y|Z)$$
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is the same under P and $\tilde{P}$.

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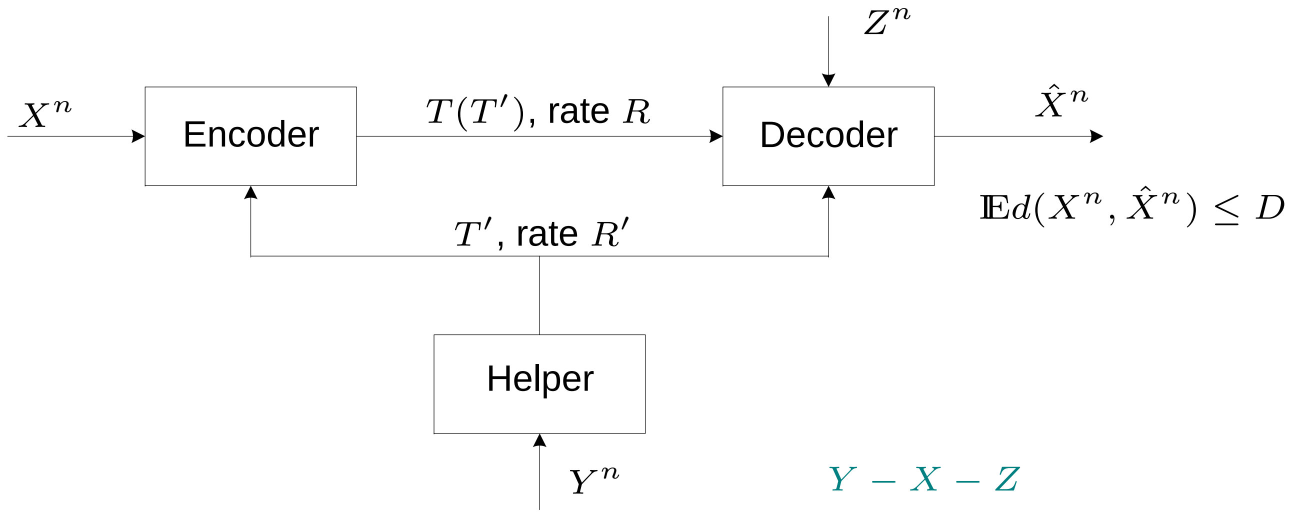
is the same under P and $\tilde{P}$.

Thus, it is enough to prove:

Achievability for any $P_{X,Y} \cdot P_{Z|X} \cdot P_{U|Y} \cdot P_{V|U,X}$

Upper bound for $P_{X,Y} \cdot P_{Z|X} \cdot P_{U|Y} \cdot P_{V|U,X,Y}$

SI Utilization



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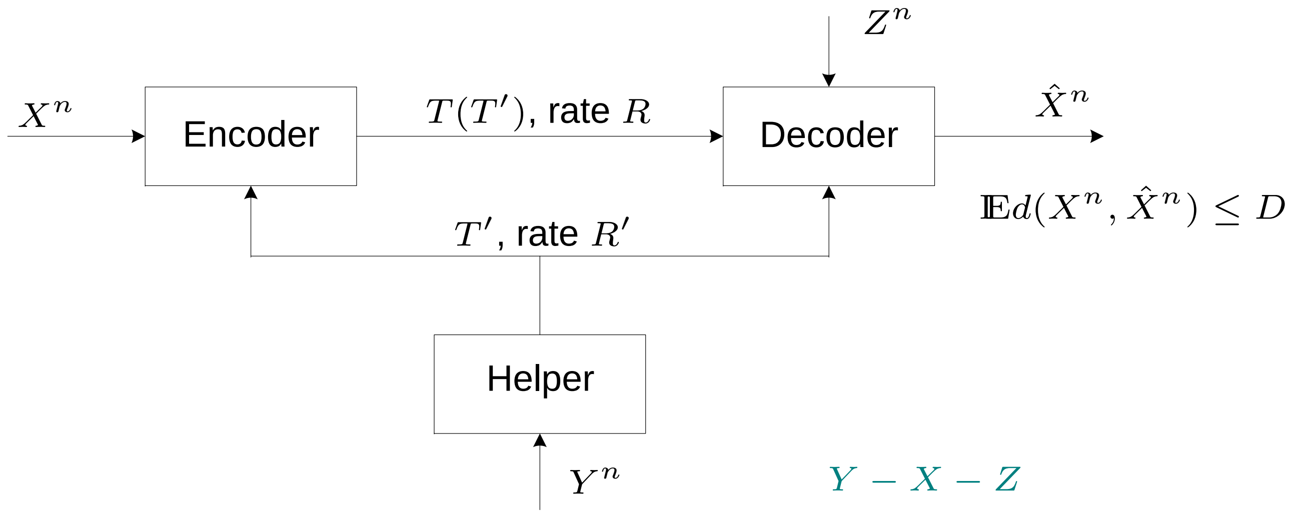
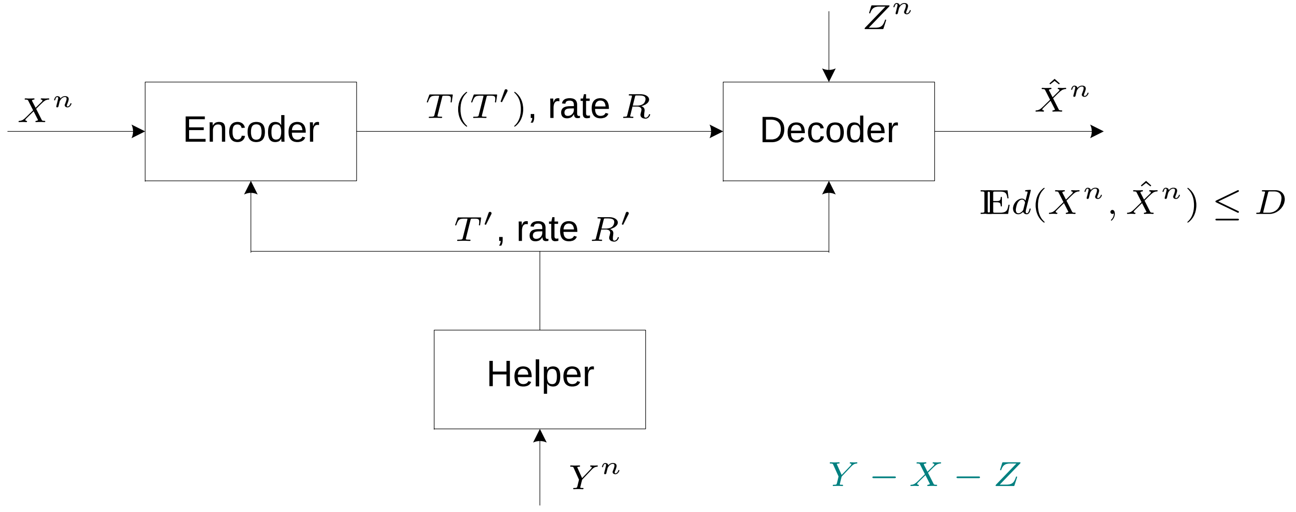


Due to $Y - X - Z$, X is a better SI than Z, in coding for Y (helper).

SI Utilization



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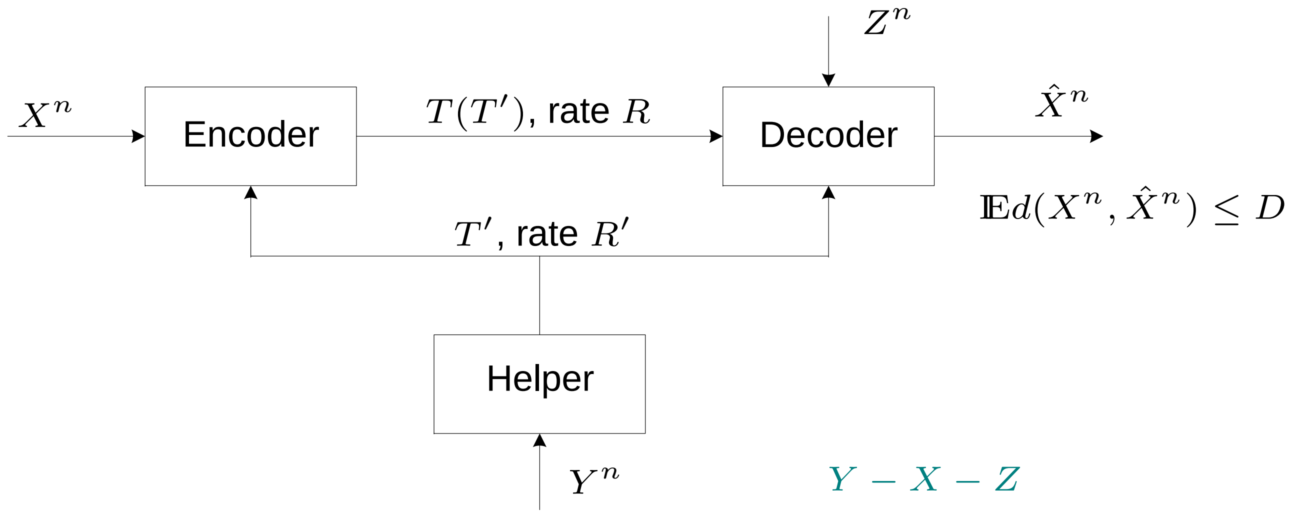
$\Rightarrow$ Would expect

$$R' \ge I(U;Y|Z) + I(W;Y|U,X), \qquad (U,W) - Y - X.$$

“Rate distortion when side information may be absent," Heegard & Berger, 1985.

SI Utilization



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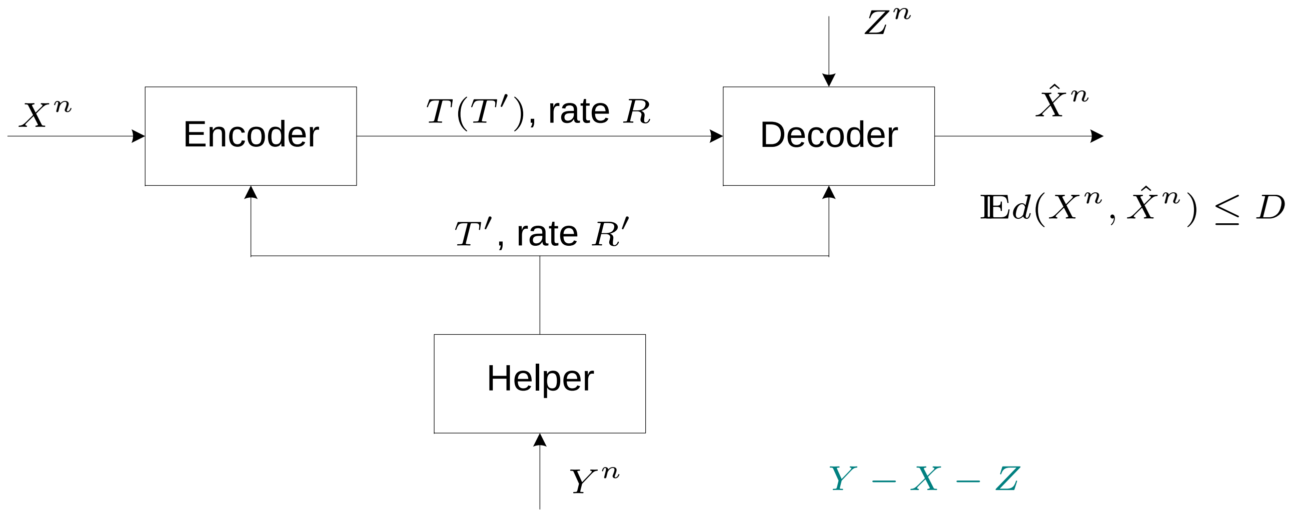
Yet, our result says: $R' \geq I(U;Y|Z)$

SI Utilization



$Y - X - Z$

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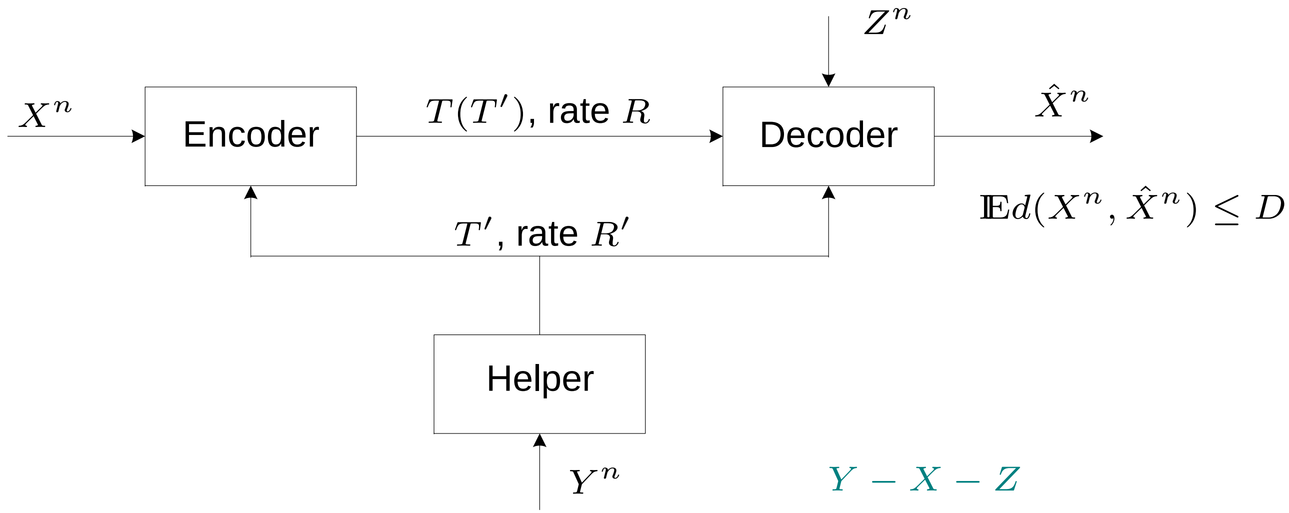

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Conclusion:

When sending a common message to the encoder and decoder, the source encoder does not use the extra SI X^n.

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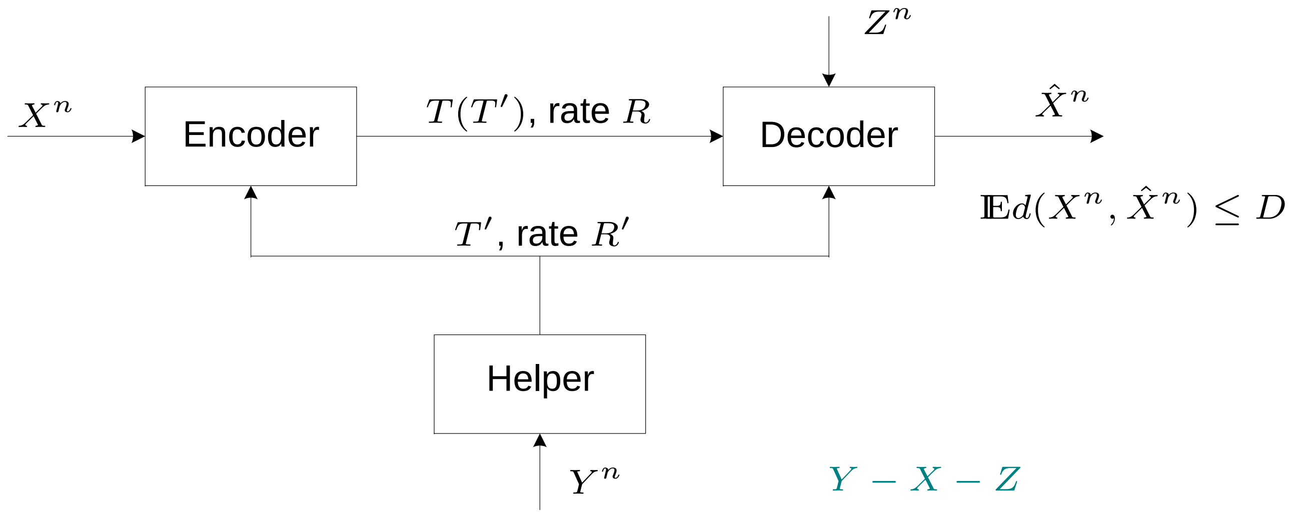

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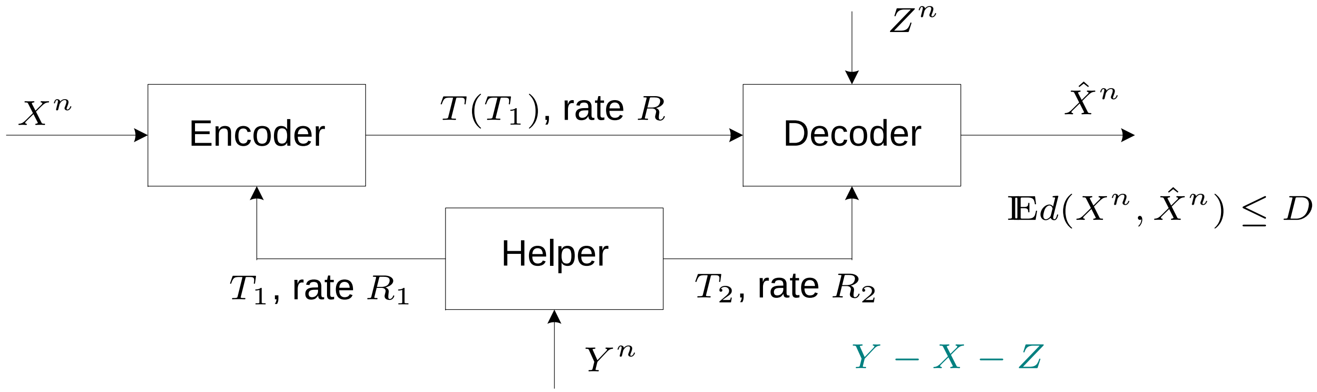
Conclusion:
When sending a common message to the encoder and decoder, the source encoder does not use the extra SI X^n.

Neither would he use extra rate.

Independent Rates With SI - Main Result



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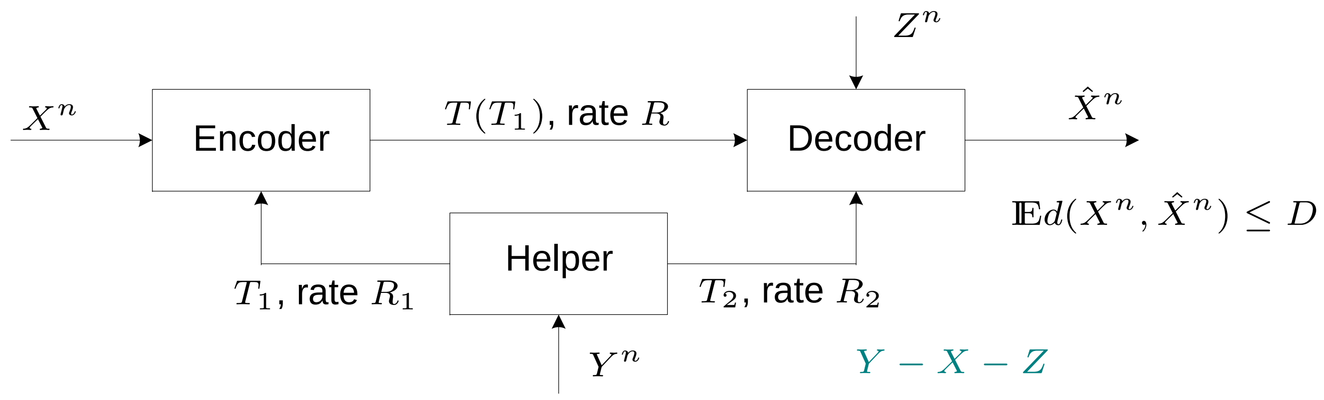

Theorem 4 Let $R_1 > R_2$, and let the side information Z satisfy $Y - X - Z$. For any code with independent helper rates R_1, R_2, there exists a code with common helper rate R_2, with essentially the same performance (the same R and D).

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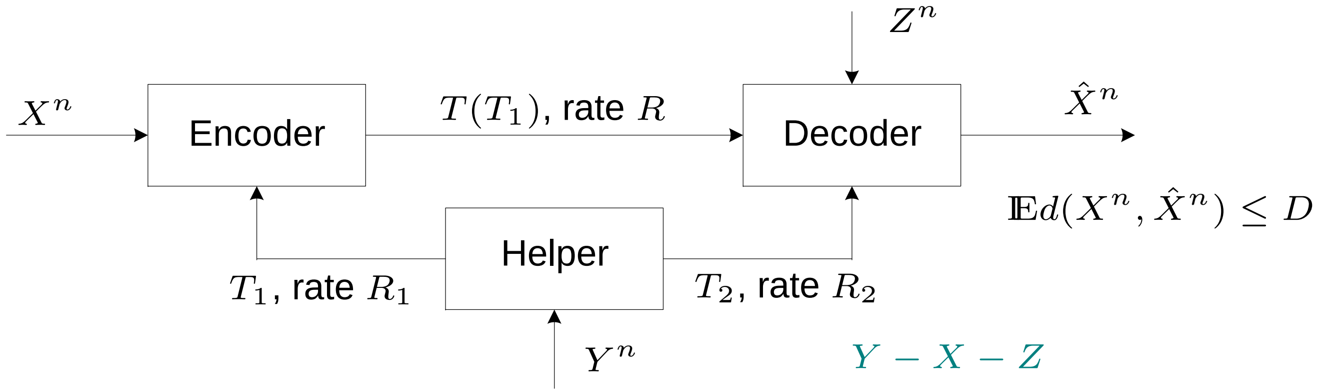

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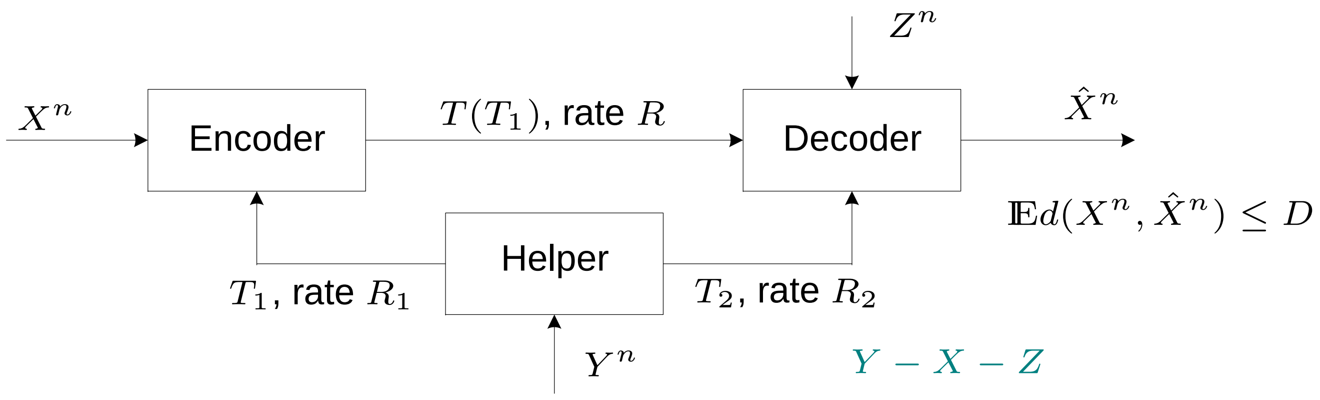

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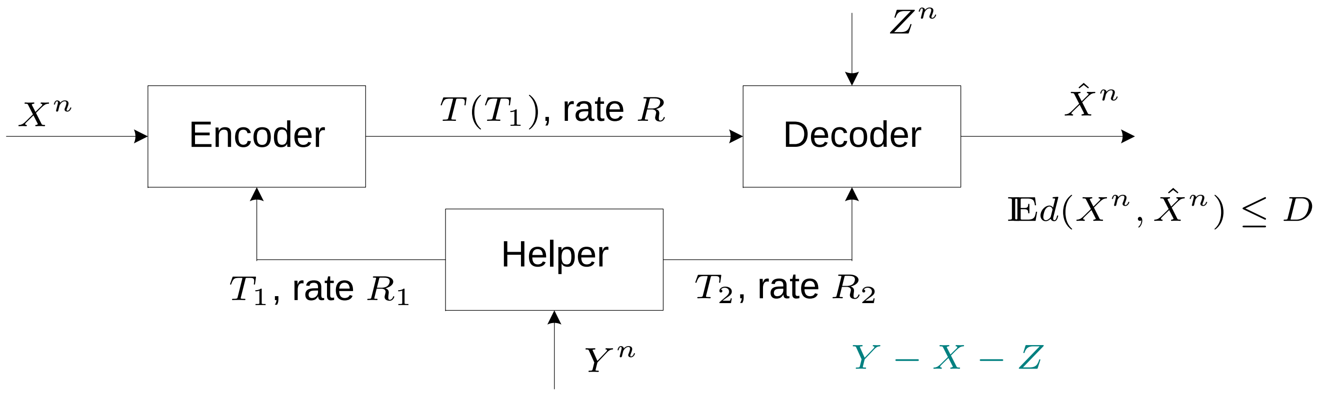
Proof – Operational arguments.

Extra helper rate to the encoder is not used. Moreover, even when independent messages are allowed, sending a common helper message yields optimal performance. $(R_1 \geq R_2)$.

Independent Rates With SI (cont'd)

What about reducing R_1 below R_2?



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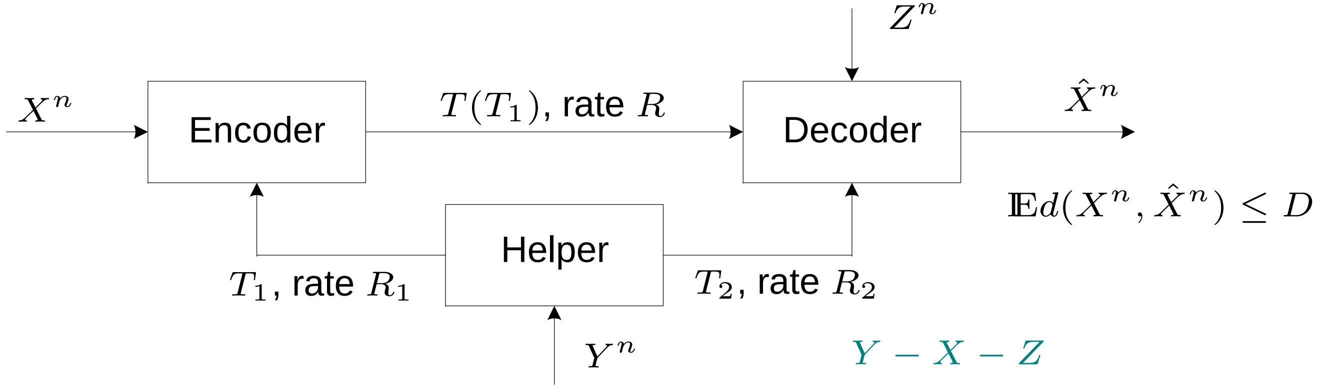

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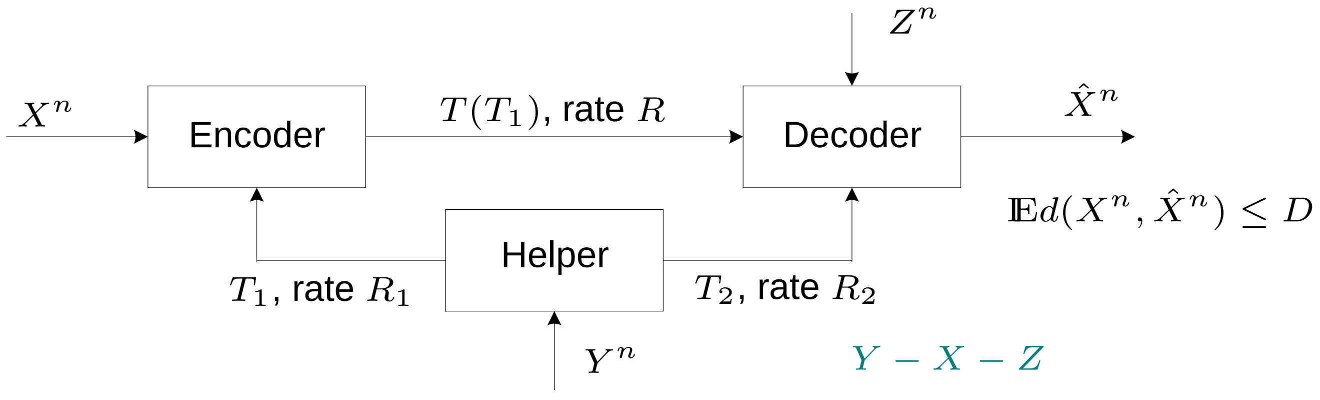

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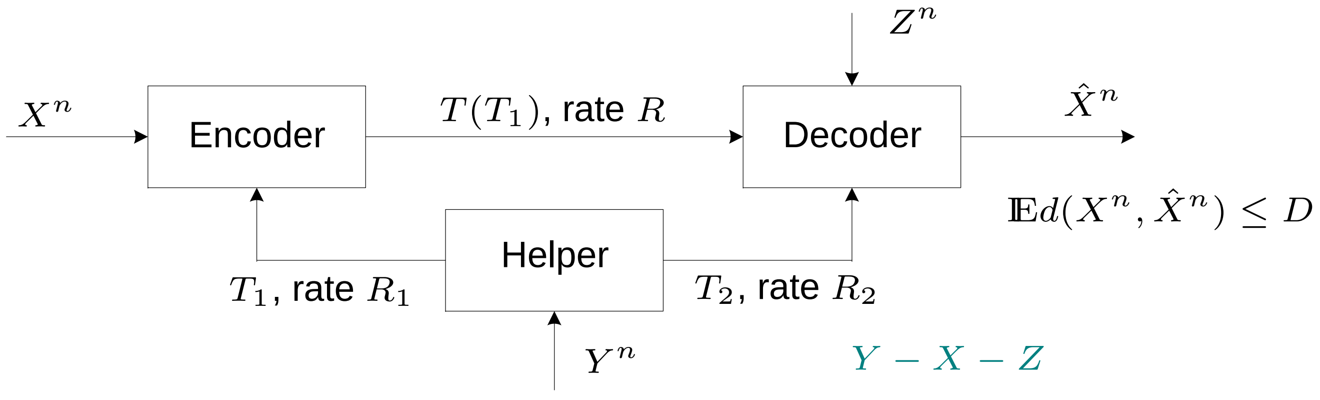

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Main idea – use binning w.r.t X to reduce the *rate* to the encoder (but not to improve the quality of $\hat{Y}^n$)

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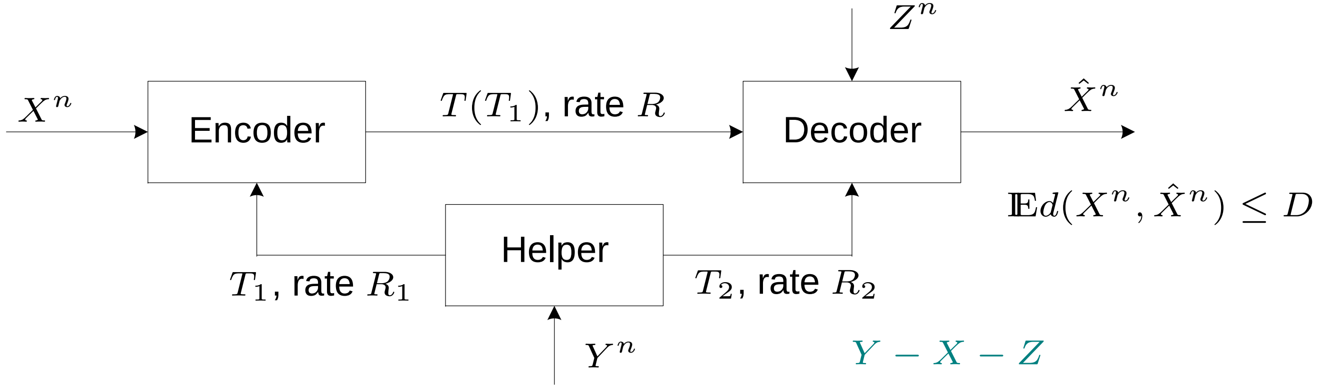

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$$

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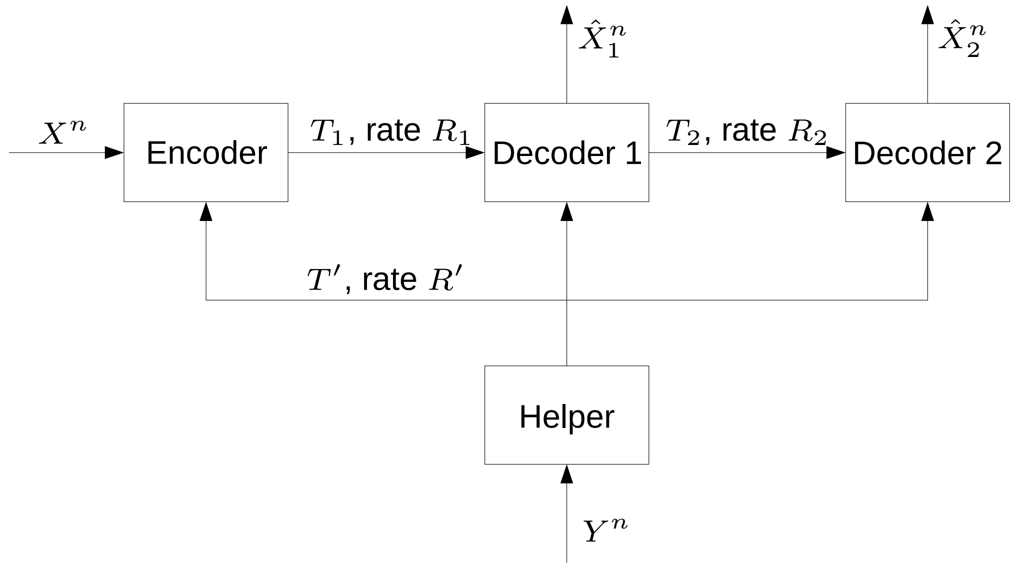
$$
\begin{aligned}
R_1 &\geq I(U;Y|Z) - I(U;X|Z) \\
&= I(U;Y) - I(U;X)
\end{aligned}
$$

Due to $U - Y - X - Z$, R_1 is strictly less than R_2.

Cascade Rate-Distortion With a Helper



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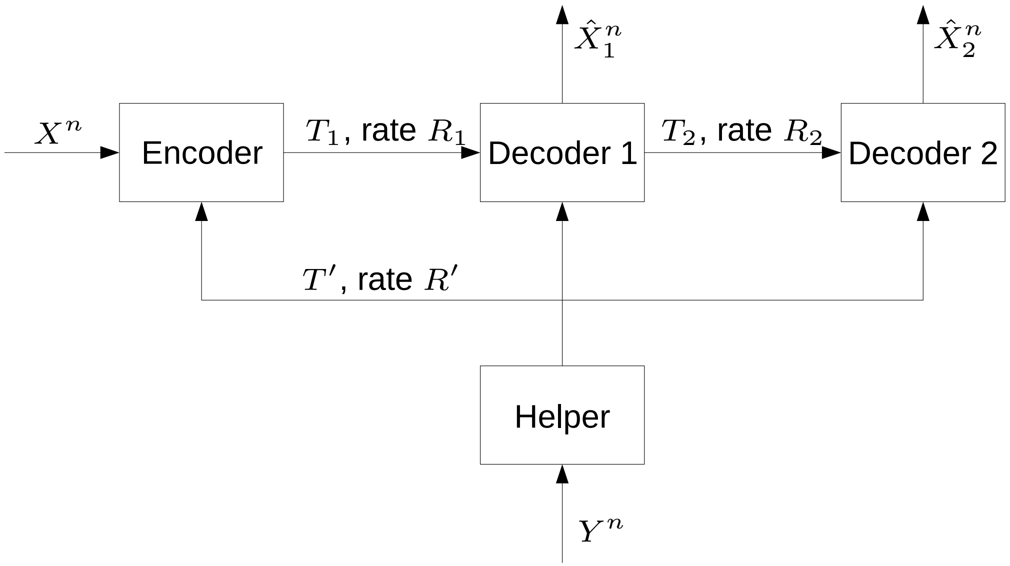
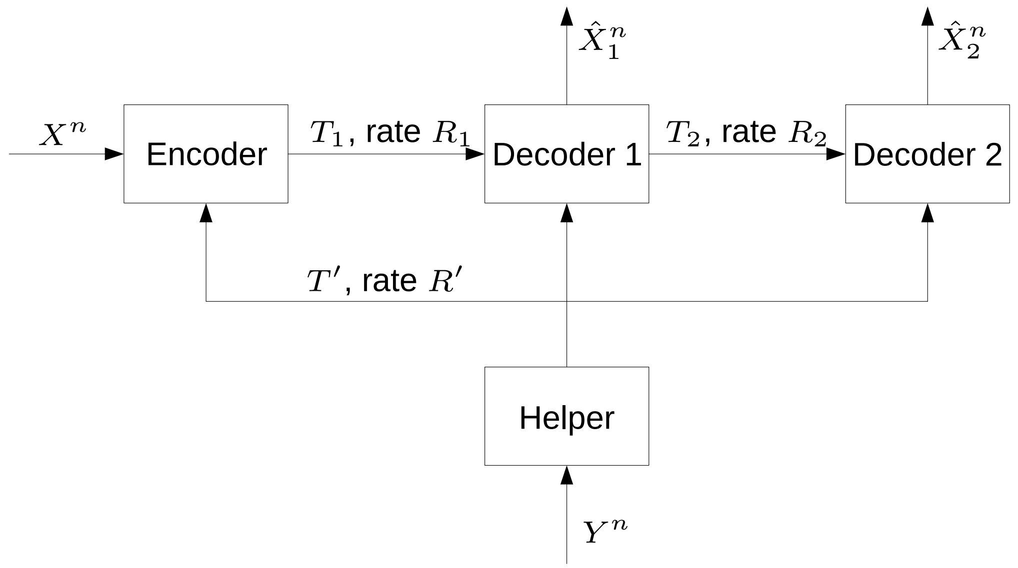


- The definitions and communication protocol are as before, with

$$\hat{X}_i^n = \hat{X}_i^n(T', T_i), \qquad i = 1, 2,$$

Cascade Rate-Distortion With a Helper



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$$\mathbb{E} d_i(X^n, \hat{X}_i^n) \leq D_i, \quad i = 1, 2.$$

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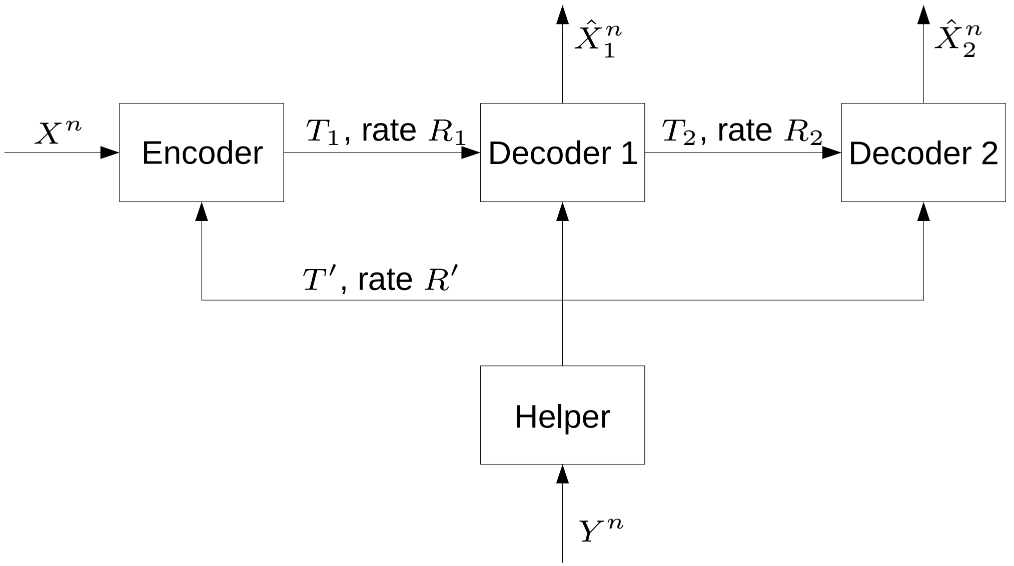

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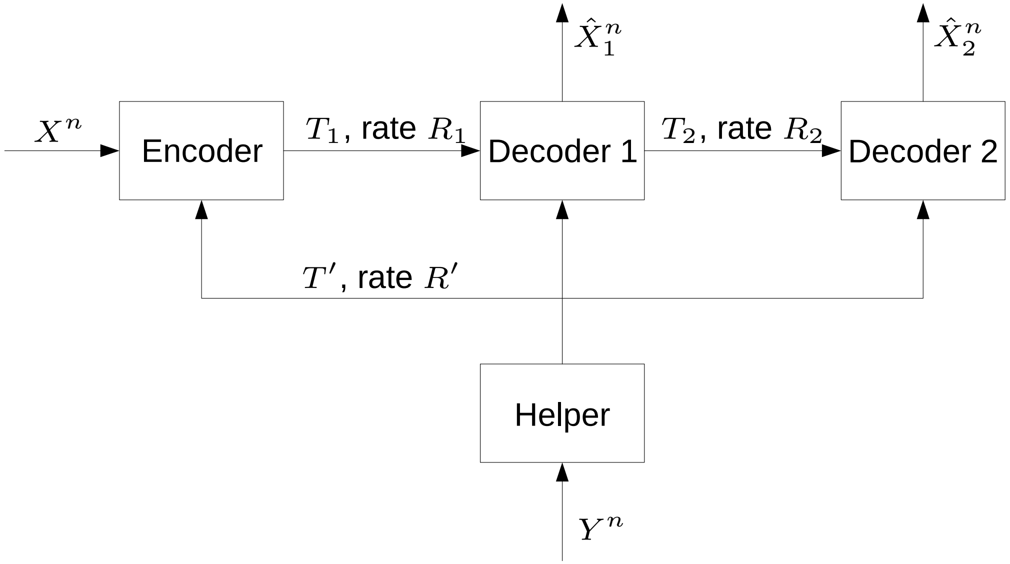
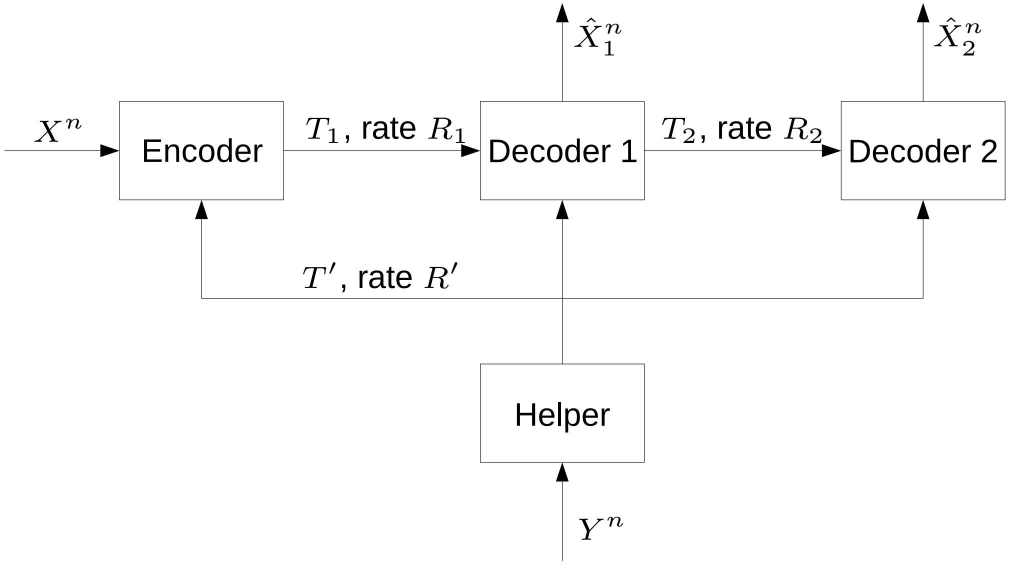
- the rate distortion region is denoted $\mathcal{R}_c(D_1, D_2)$.

Main Result


$\hat{X}_1^n$
$\hat{X}_2^n$
X^n
Encoder
T_1, rate R_1
Decoder 1
T_2, rate R_2
Decoder 2
T', rate R'
Helper
Y^n

Main Result



Theorem 5 $(R', R_1, R_2) \in \mathcal{R}_c(D_1, D_2)$ if and only if

$$\begin{aligned} R' &\geq I(Y;V) \\ R_1 &\geq I(X;\hat{X}_1,\hat{X}_2|V) \\ R_2 &\geq I(X;\hat{X}_2|V) \\ D_i &\geq \mathbb{E}d(X,\hat{X}_i), \quad i=1,2 \end{aligned}$$

for some joint distribution of the form

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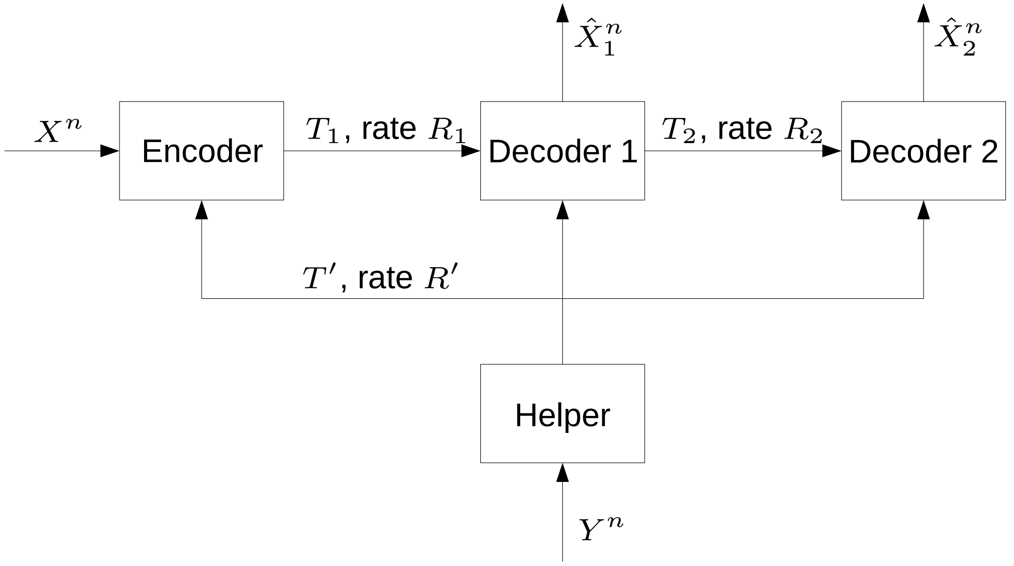

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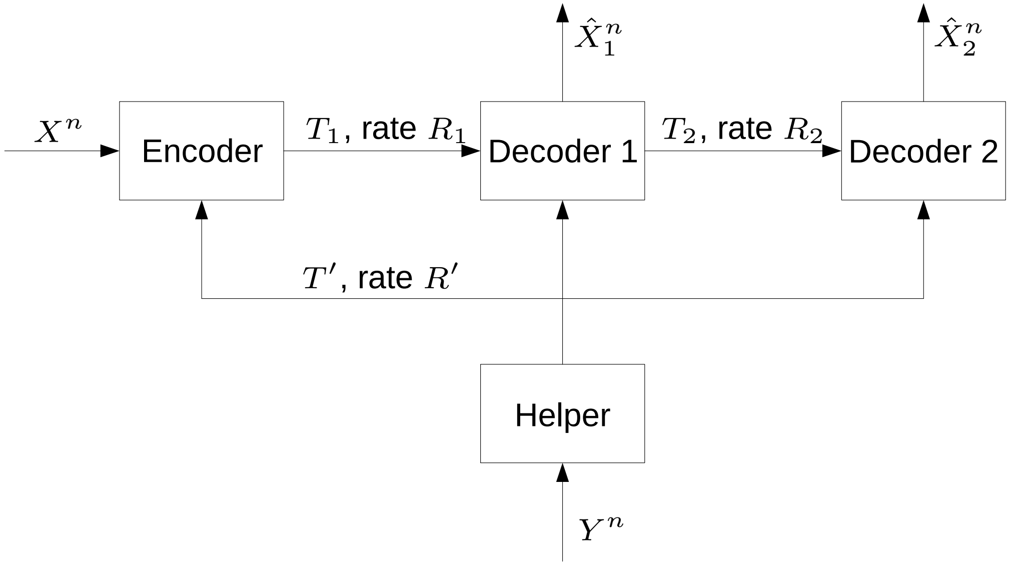

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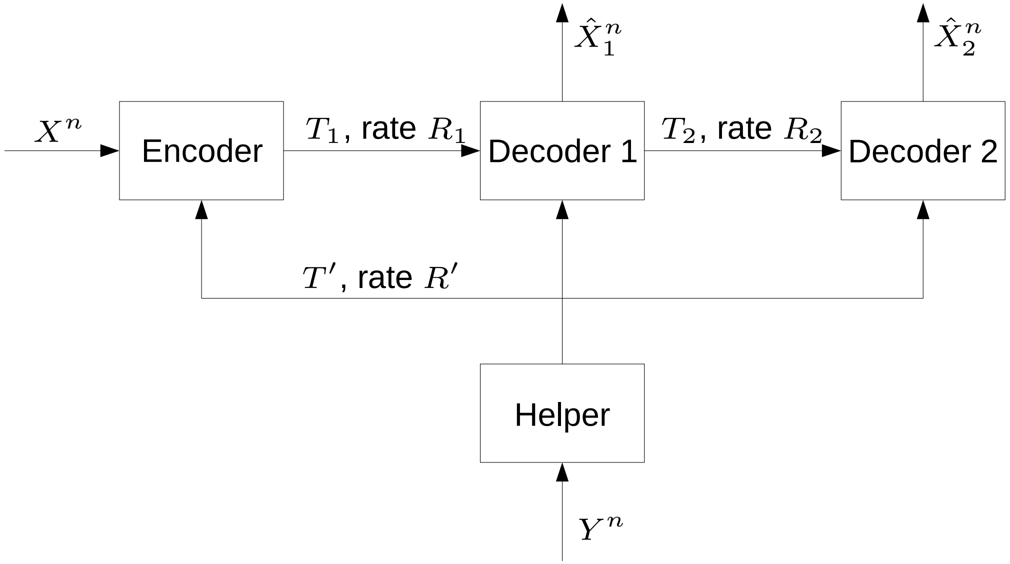
$$\begin{aligned} R' &\geq I(Y;V) & & V - Y - X \\ R_1 &\geq I(X;\hat{X}_1,\hat{X}_2|V) & & Y - (V,X) - (\hat{X}_1,\hat{X}_2) \\ R_2 &\geq I(X;\hat{X}_2|V) & & \\ D_i &\geq \mathbb{E}d(X,\hat{X}_i), \quad i=1,2 & & \end{aligned}$$

for some joint distribution of the form

$$P_{X,Y} \cdot P_{V|Y} \cdot P_{U|Y} \cdot P_{\hat{X}_1,\hat{X}_2|V,X,Y}.$$

Main Result



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Summary

- Solved the problem of rate-distortion with helper and SI at the decoder, for the case of common helper message.
- Implies a solution to the case of independent helper messages, with $R_1 \geq R_2$.
- Can reduce slightly the helper rate to the encoder (R_1), and get the same performance as with $R_1 \geq R_2$ (but this depends on external random variables).
- Solved the model of cascade rate-distortion with a helper. Here helper must send common messages; cannot play with rates.
- Have also a solution for the two way problem with a helper.

Thank You!