The Multiple Access Channel with Causal and Strictly Causal Side Information at the Encoders

Amos Lapidoth and Yossef Steinberg

Outline

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- Problem Formulation: The MAC with strictly causal and causal common SI

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- An achievable region for the strictly causal model

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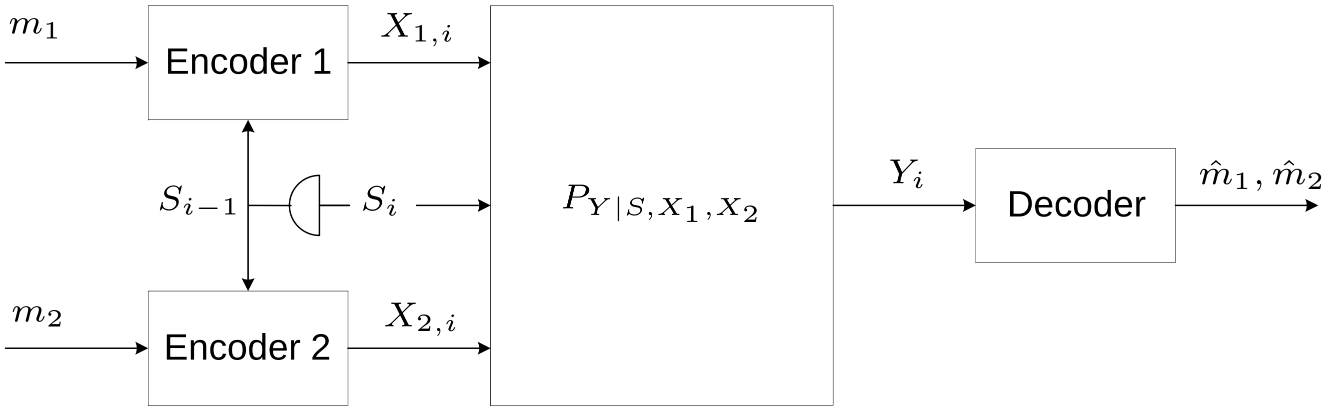
- Example
- Two independent states

Problem Formulation

MAC with strictly causal side information (SI):



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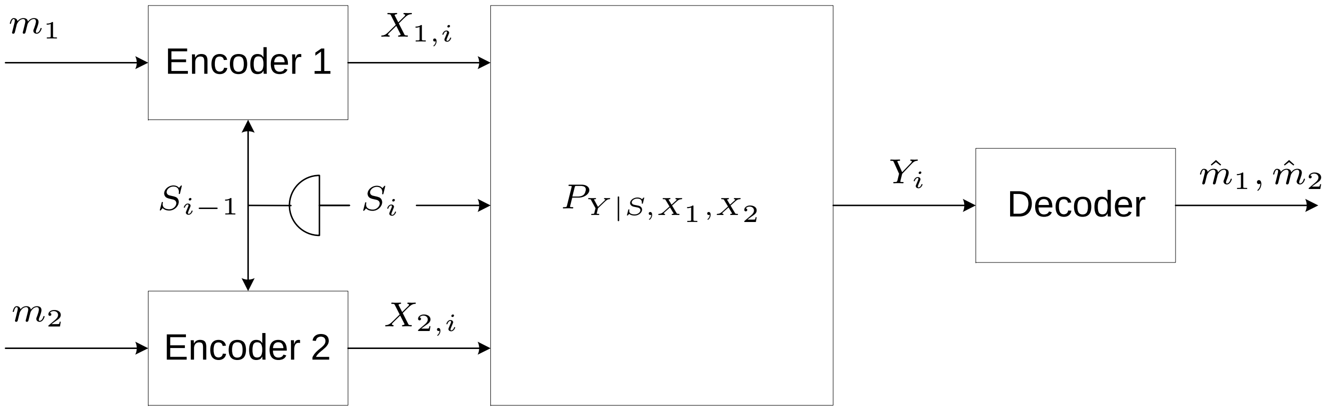

- One state sequence S^n, available to the encoders in a strictly causal manner:

$$X_{1,i} = f_{1,i}(m_1, S^{i-1}), \quad X_{2,i} = f_{2,i}(m_2, S^{i-1}), \quad i = 1, \ldots, n$$

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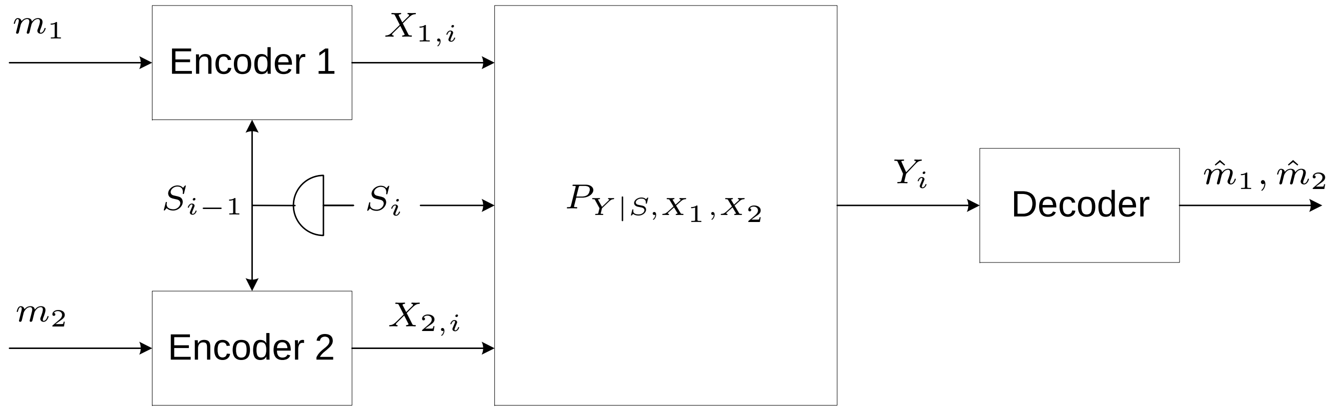

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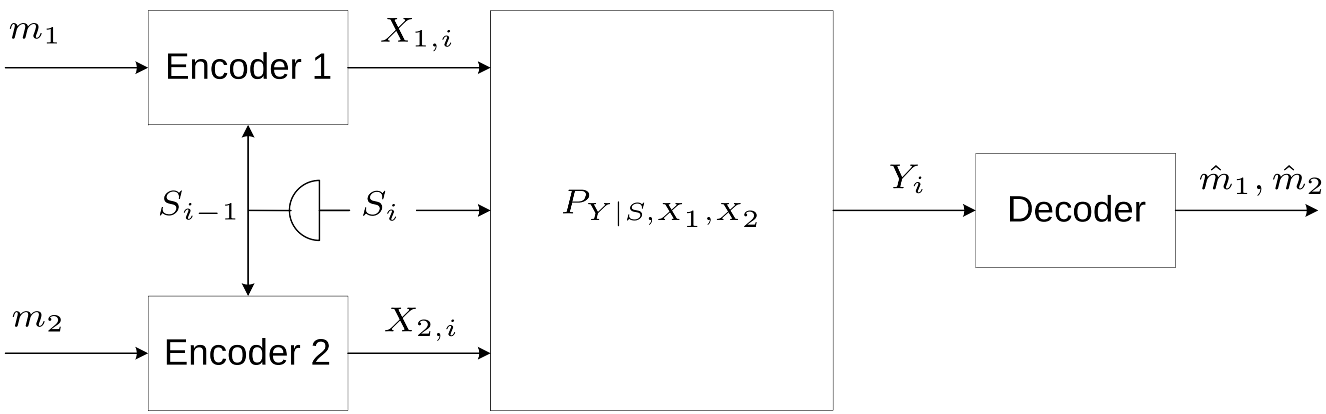

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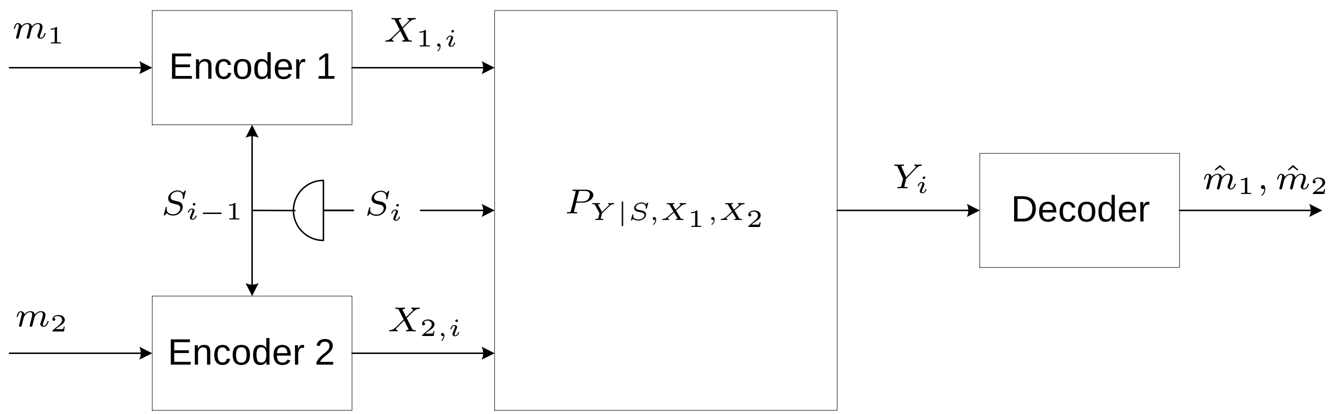

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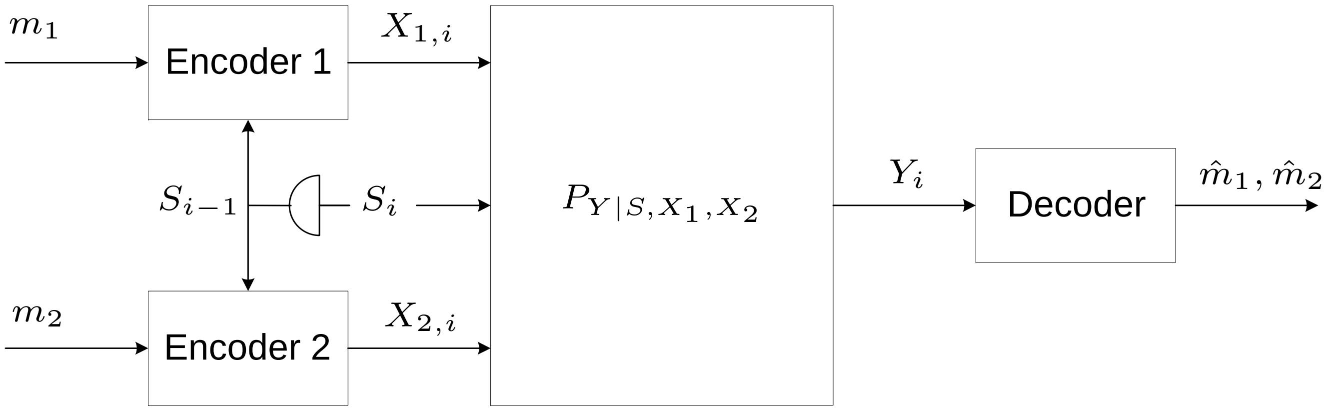

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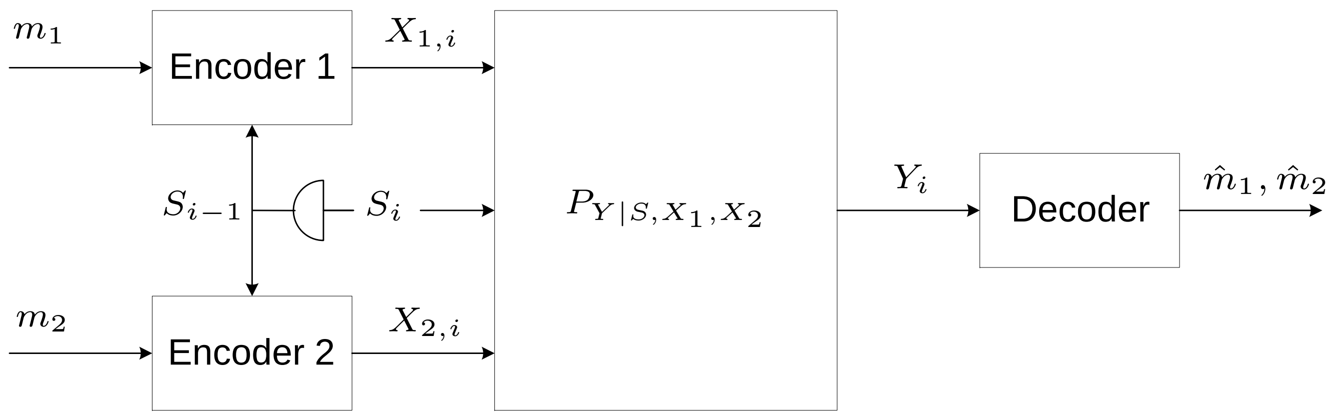

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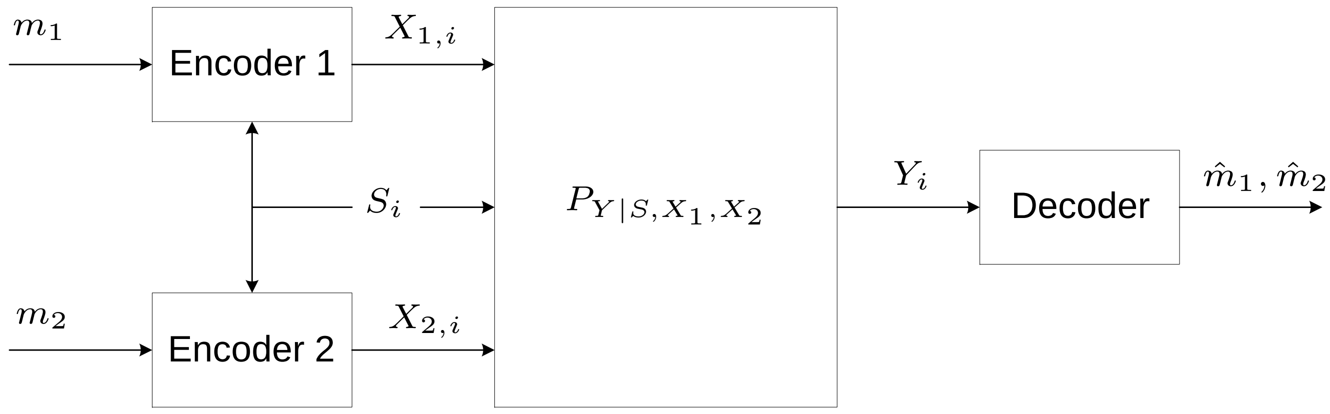

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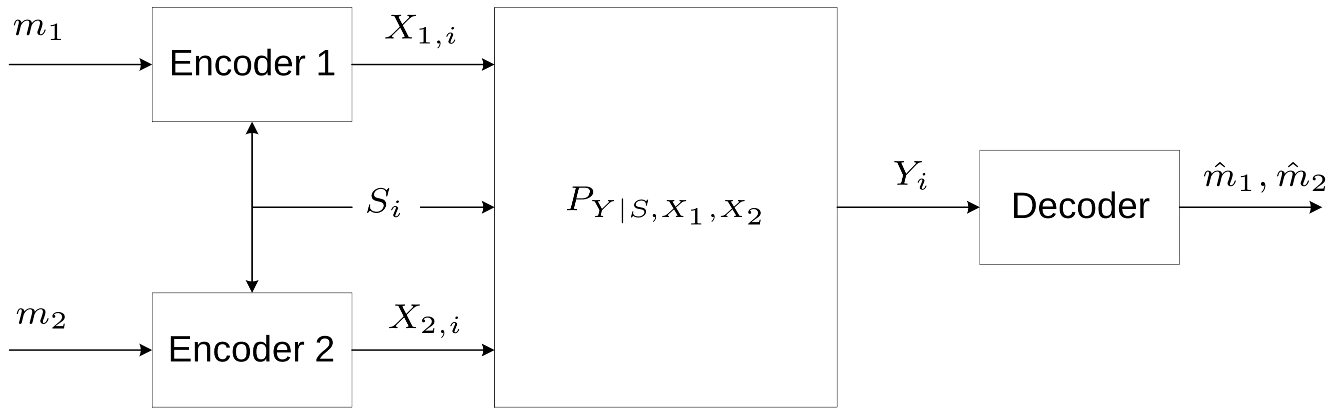

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\begin{aligned}
nR - n\epsilon_n &\leq I(M;Y^n) = \sum_{i=1}^{n} I(M;Y_i|Y^{i-1}) \\
&\leq \sum_{i=1}^{n} I(M,Y^{i-1};Y_i) \\
&\leq \sum_{i=1}^{n} I(M,Y^{i-1},X_i;Y_i) \\
&= \sum_{i=1}^{n} I(X_i;Y_i) \\
&\leq \max_{P_X} I(X;Y) = nC
\end{aligned}
$$

where C is the capacity without SI.

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- Strictly causal SI does not increase the capacity of the single user channel (a reminiscent of the situation in feedback capacity)
- Transmission of the state (or compressed version thereof) to the other side is sub optimal: waste of precious rate, without increase in capacity.

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- In the MAC: The two users can *cooperate* in transmitting the noise (state) to the decoder.

MAC with SC SI

$\mathcal{R}_{\text{s-c}}$ - the CH of all $(R_1, R_2, \Gamma_1, \Gamma_2)$ satisfying

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\begin{aligned}
R_1 &\leq I(X_1;Y|X_2,U,V) \\
R_2 &\leq I(X_2;Y|X_1,U,V) \\
R_1 + R_2 &\leq I(X_1,X_2;Y|U,V) \\
R_1 + R_2 &\leq I(X_1,X_2,V;Y) - I(V;S) \\
\Gamma_k &\geq \mathsf{E}[\phi_k(X_k)], \qquad k = 1,2
\end{aligned}
$$

for some joint distribution

$$P_{U,V,X_1,X_2,S,Y} = P_S P_{X_1|U} P_{X_2|U} P_U P_{V|S} P_{Y|S,X_1,X_2}.$$

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Main result

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Theorem 1 (Strictly-Causal SI) $\mathcal{R}_{\text{s-c}} \subseteq \mathcal{C}_{\text{s-c}}$

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- The two codes are decoupled.

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- The decoded state of block B is used to decode the messages sent at block B: private messages, and compressed state of block $B-1$....

Example

The Gaussian MAC where the state comprises the channel noise

$$Y = X_1 + X_2 + S, \qquad S \sim \mathcal{N}\left(0, \sigma_s^2\right)$$

$$\mathsf{E}\left[X_1^2\right] \leq \Gamma_1, \qquad \mathsf{E}\left[X_2^2\right] \leq \Gamma_2.$$

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I.e., the full cooperation line can be achieved.

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I.e., the full cooperation line can be achieved.

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The Gaussian MAC where the state comprises the channel noise

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- A Schalkwijk-Kailath algorithm

MAC with causal SI

The region we had for the strictly causal case is still achievable

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R_1 &\leq I(X_1; Y | X_2, U, V) \\
R_2 &\leq I(X_2; Y | X_1, U, V) \\
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R_1 + R_2 &\leq I(X_1, X_2, V; Y) - I(V; S) \\
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$\Rightarrow$ Use Shannon strategies on top of our block Markov scheme.

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But now, X_1, X_2 can depend on S.

Replace (X_1,X_2) by (U_1,U_2) independent of S, and let

$$P_{X_1|U,U_1,S}, \quad P_{X_2|U,U_2,S}$$

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$\mathcal{R}_{\text{cau}}$ - the CH of all $(R_1, R_2, \Gamma_1, \Gamma_2)$ satisfying

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Theorem 2 (Causal SI) $\mathcal{R}_{\text{cau}} \subseteq \mathcal{C}_{\text{cau}}$

The naïve approach

The naïve approach – using Shannon strategies, without block Markov coding of the state.

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Q is a time sharing random variable, and

$$P_{Y|T_1,T_2}(y|t_1, t_2) = \sum_{s \in \mathcal{S}} P_S(s) P_{Y|S,X_1,X_2}\big(y|s, t_1(s), t_2(s)\big).$$

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We denote this region as $\mathcal{R}^{\text{naïve}}$.

The naïve approach

- $\mathcal{R}_{\text{cau}}$ contains the region of the naïve approach, since we can always choose degenerate V.

- In some cases, the inclusion is strict.

Example

The noiseless binary MAC with input selector:

$$\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y} = \{0,1\}, \quad \mathcal{S} = \{1,2\}, \quad P_S(S=2) = p > 0.5$$

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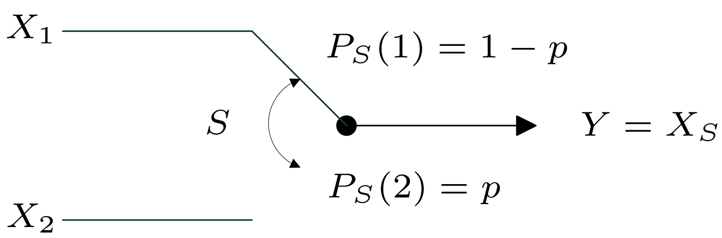

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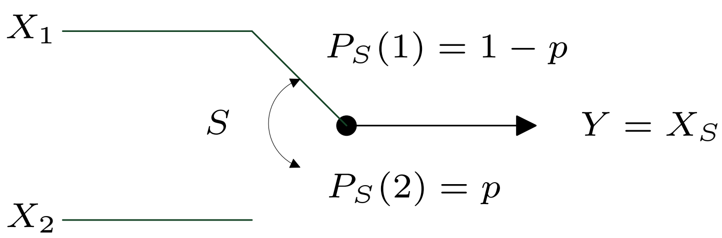
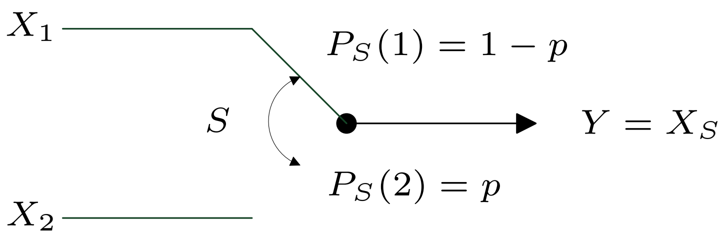


- If the decoder knows S, user 1 can transmit at rate $1-p$.
- Hence, $1-p$ is an upper bound on the transmission rate of user 1 in our model.

Example



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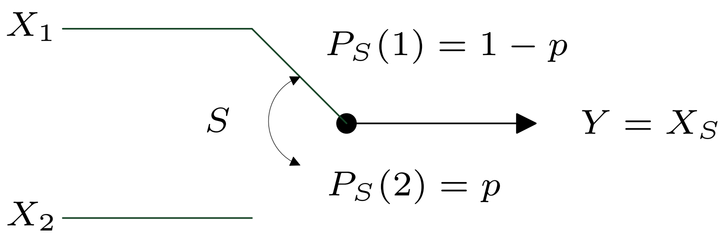

With a proper choice of random variables in $\mathcal{R}_{\text{cau}}$

$$(R_1, R_2) = \Big(\min\big\{1-p, 1-H_b(p)\big\},\, 0\Big) \in \mathcal{R}_{\text{cau}}$$

(Observe – achieves the maximal rate of user 1 for $p \geq H_b(p)$.)

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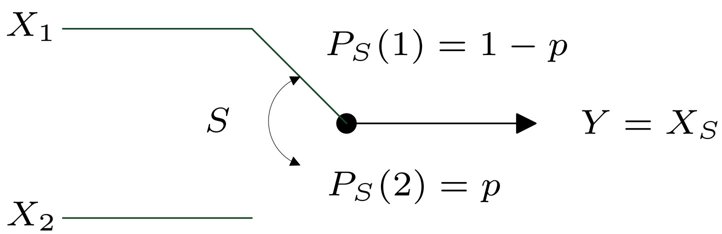

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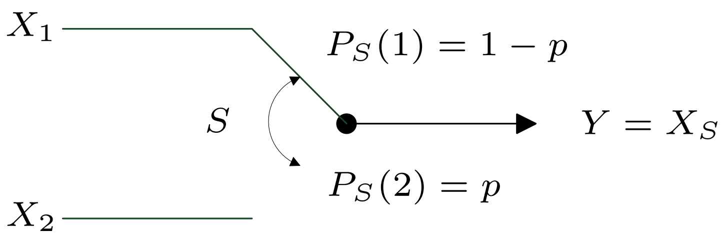

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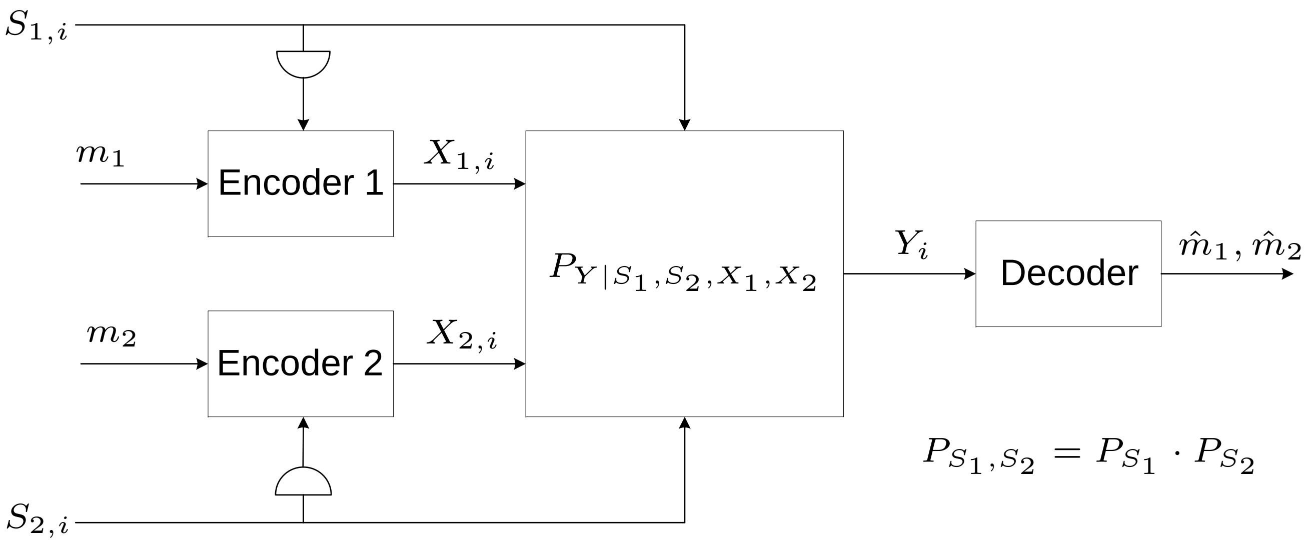
For sufficiently large value of p,

$$R_{2,\text{max}}^{\text{naïve}} < \min\big\{1 - p, 1 - H_b(p)\big\}$$

MAC with independent SI streams



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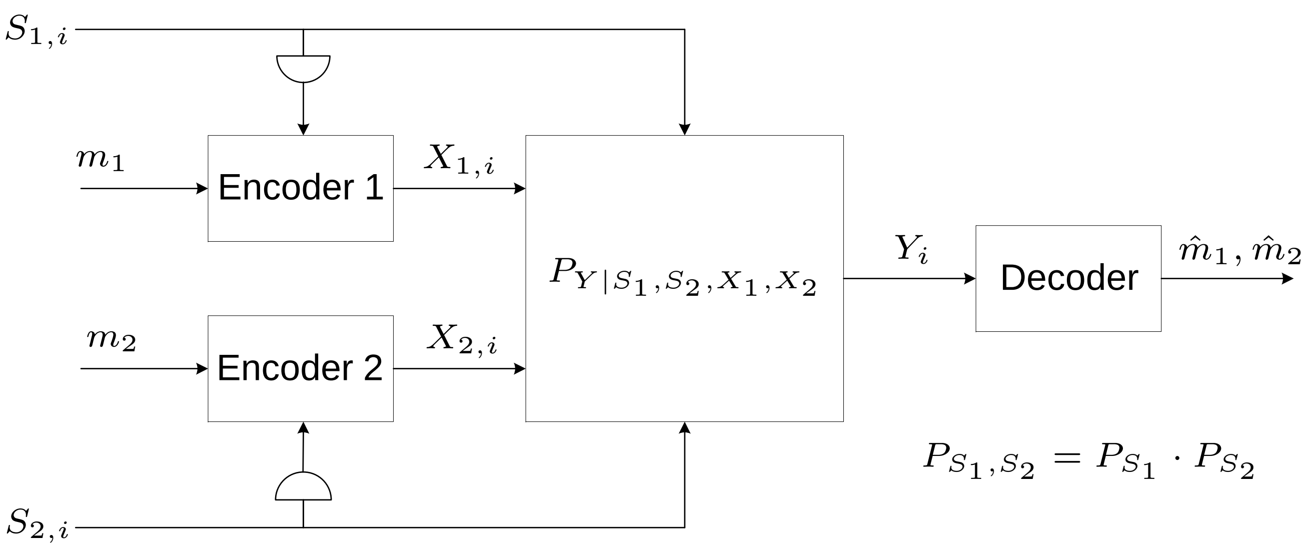

- *Cooperation* in the compression and transmission of the state is not possible.

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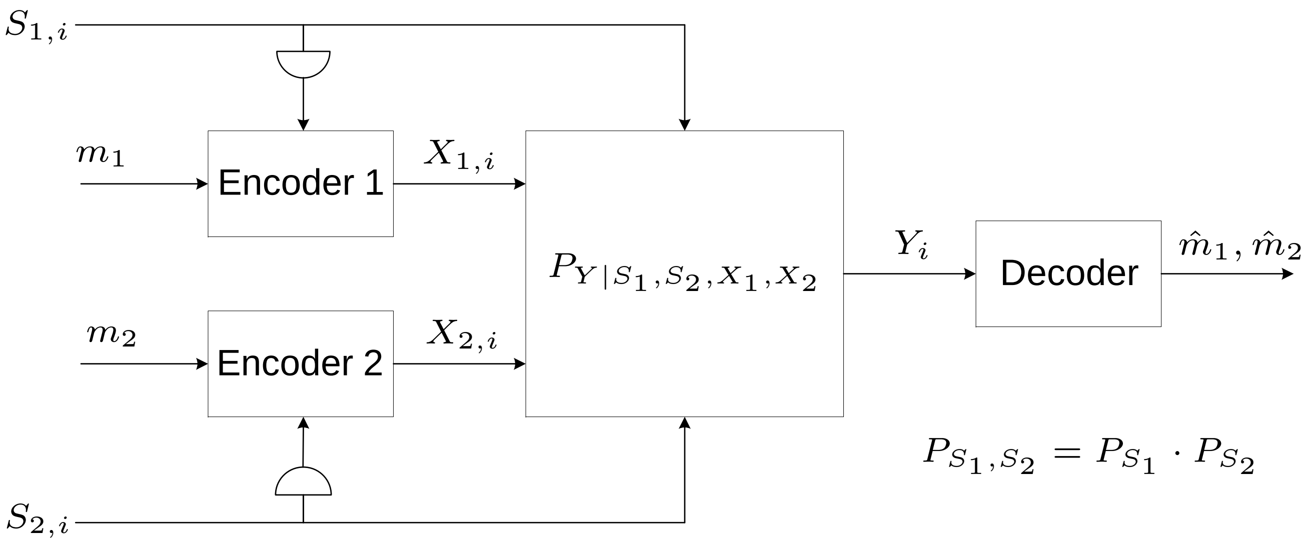

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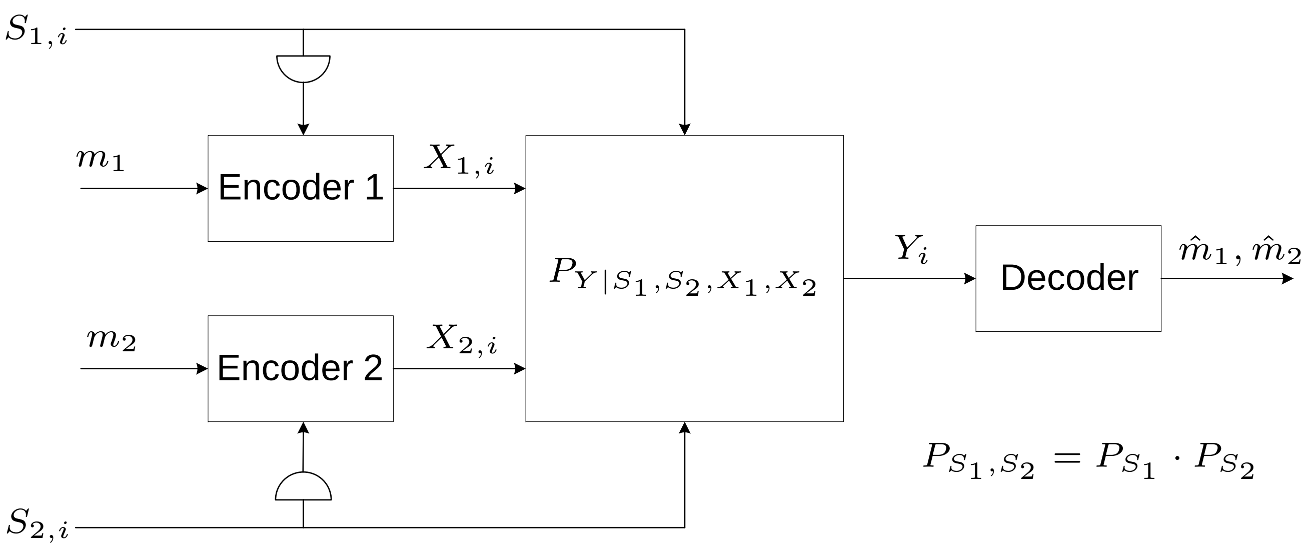

- *Cooperation* in the compression and transmission of the state is not possible.
- Yet, compression and transmission of the states to the decoder is beneficial, and enlarges the capacity region of the MAC.
- Utilize distributed Wyner-Ziv compression and block Markov coding (ISIT 2010).

Summary

- Derived achievable region for the MAC with common strictly causal SI, based on block Markov encoding of the state.
- Strictly causal SI enlarges the capacity region of the MAC.
- Extended the results to causal SI
- The new region for causal Si is strictly better that the region obtained by the naïve approach.

- Strictly causal SI is beneficial even when the states available at the encoders are independent (ISIT 2010).

Thank You!