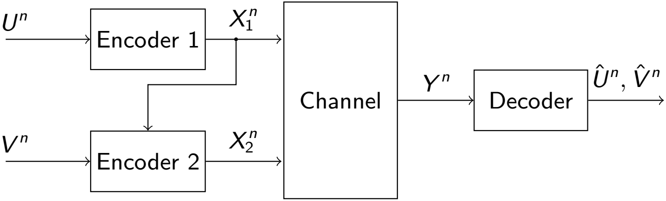
The Multiple Access Channel with Correlated Sources and Cribbing Encoders

Eliron Amir and Yossef Steinberg

IZS 2012

Problem formulation



- Encoders:

$$X_{1,i} = f_{1,i}(U^n)$$
$$X_{2,i} = f_{2,i}(V^n, X_1^{i-1}) \quad \text{(strictly causal cribbing)},$$

- Memoryless channel $P_{Y|X_1,X_2}$ and source $(U, V) \sim P_{U,V}$
- Lossless transmission of U^n and V^n.

Transmissibility conditions - ?

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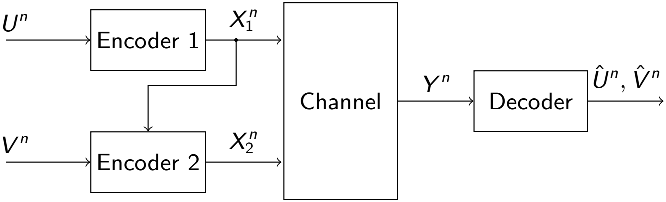

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Previous work

- Willems & van der Meulen, 85: MAC with cribbing encoders

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  - Two users MAC, three independent sources (U_0, U_1, U_2), U_0 known to both users.

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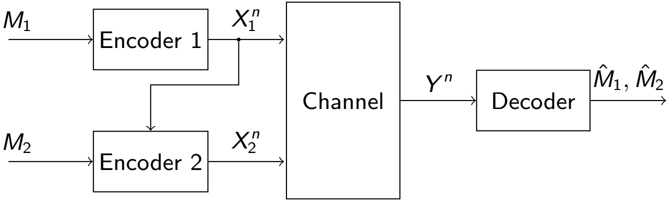
- Slepian & Wolf, Sept. 73: Multiple access channel with correlated sources:
  - Two users MAC, three independent sources (U_0, U_1, U_2), U_0 known to both users.
- Cover, El Gamal, & Salehi, 80: Multiple access channels with arbitrarily correlated sources

Previous results

MAC with cribbing encoders, W&M, 85:

W&M studied all possible cribbing combinations:



One sided, strictly causal cribbing

$$X_{1,i} = f_{1,i}(M_1)$$
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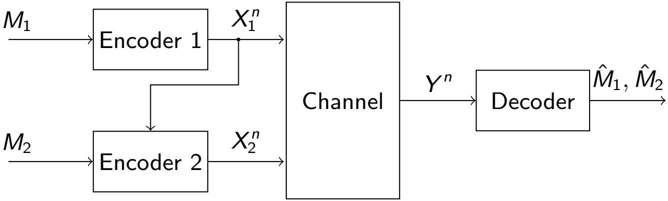

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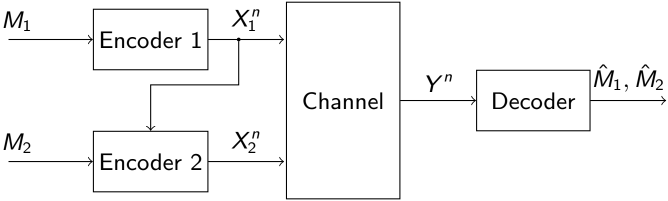

One sided, non-causal cribbing

$$X_{1,i} = f_{1,i}(M_1)$$
$$X_{2,i} = f_{2,i}(M_2, X_1^n)$$

Previous results

MAC with cribbing encoders, W&M, 85:

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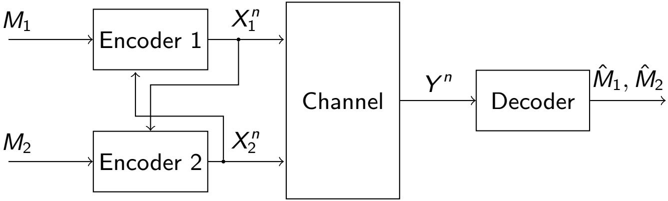

Two sided, stricly causal

$$X_{1,i} = f_{1,i}(M_1, X_2^{i-1})$$
$$X_{2,i} = f_{2,i}(M_2, X_1^{i-1})$$

Previous results

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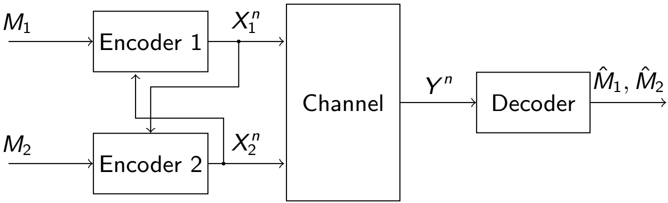

Two sided, causal

$$X_{1,i} = f_{1,i}(M_1, X_2^{i-1})$$
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Previous results

MAC with cribbing encoder, W&M 85

Main theme: cribbing allows dependence between the MAC inputs:

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Main theme: cribbing allows dependence between the MAC inputs:

- The capacity region for one sided strictly causal cribbing

$$
\begin{aligned}
R_1 &\leq H(X_1|W) \\
R_2 &\leq I(X_2;Y|X_1,W) \\
R_1 + R_2 &\leq I(X_1,X_2;Y)
\end{aligned}
$$

for some $P_W P_{X_1|W} P_{X_2|W}$.

Previous results

MAC with cribbing encoder, W&M 85

Main theme: cribbing allows dependence between the MAC inputs:

- The capacity region for one sided strictly causal cribbing

$$\begin{aligned} R_1 &\le H(X_1|W) \\ R_2 &\le I(X_2;Y|X_1,W) \\ R_1+R_2 &\le I(X_1,X_2;Y) \end{aligned}$$

  for some $P_W P_{X_1|W} P_{X_2|W}$.

- The capacity region for one sided causal/non-causal cribbing

$$\begin{aligned} R_1 &\le H(X_1) \\ R_2 &\le I(X_2;Y|X_1) \\ R_1+R_2 &\le I(X_1,X_2;Y) \end{aligned}$$

  for some P_{X_1,X_2}.

Previous results

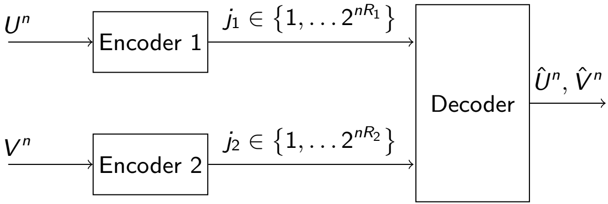
MAC without cribbing

$$
\begin{aligned}
R_1 &\leq I(X_1; Y|X_2, Q) \\
R_2 &\leq I(X_2; Y|X_1, Q) \\
R_1 + R_2 &\leq I(X_1, X_2; Y|Q)
\end{aligned}
$$

for some $P_Q P_{X_1|Q} P_{X_2|Q}$.

Previous work

Noisless coding of correlated sources, S&W 73

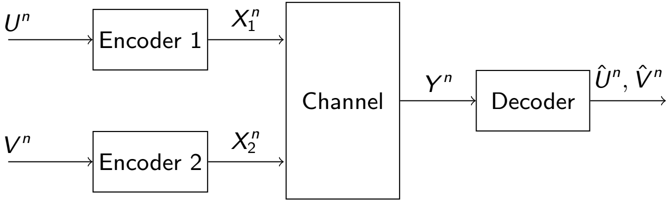


A rate pair (R_1, R_2) is achievable if and only if

$$R_1 \geq H(U|V)$$
$$R_2 \geq H(V|U)$$
$$R_1 + R_2 \geq H(U, V)$$

Previous results

Joint source-channel coding for MAC, Cover, El Gamal, & Salehi 80



The source (U, V) can be sent via the MAC with $P_e \to 0$ if

$$\begin{aligned} H(U|V) &\le I(X_1; Y|X_2, V, W) \\ H(V|U) &\le I(X_2; Y|X_1, U, W) \\ H(U, V|S) &\le I(X_1, X_2; Y|W, S) \\ H(U, V) &\le I(X_1, X_2; Y) \end{aligned}$$

where S is the *common part* $g(U) = f(V) = S$, and

$$P_{W,U,V,X_1,X_2} = P_W P_{U,V} P_{X_1|U,W} P_{X_2|V,W}.$$

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Without common part:

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where

$$P_{U,V,X_1,X_2} = P_{U,V} P_{X_1|U} P_{X_2|V}.$$

- Separation does not hold (r.h.s depends on the source)

Previous results

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- An intuitive explanation: in channel capacity problems, the messages at the inputs, M_1, M_2 are independent, resulting in $P_{X_1,X_2} = P_{X_1} P_{X_2}$. This does not fit well the source model - U and V are dependent. (??)
- Cribbing allows dependence between inputs. Does separation yield optimal performance in cribbing models?

Definitions

Strictly causal cribbing

- An (n, m, ϵ) code consists of $m+1$ encoding functions

$$f_1 : \mathcal{U}^n \to \mathcal{X}_1^m,$$

$$f_{2,i} : \mathcal{V}^n \times \mathcal{X}_1^{i-1} \to \mathcal{X}_{2,i}, \quad i = 1, 2, \ldots, m$$

  and a decoding function

$$\phi : \mathcal{Y}^m \to \mathcal{U}^n \times \mathcal{V}^n$$

  such that $P((U^n, V^n) \neq \phi(Y^m)) \leq \epsilon$.

- The rate of the code is $\rho = n/m$

Definitions
Strictly causal cribbing

(U, V) is transmissible via $P_{Y|X_1,X_2}$ at rate ρ if for every $\epsilon > 0$, $\delta > 0$, and s.l. n, there exists an $(n, n/(\rho - \delta), \epsilon)$ code for $((U, V), P_{Y|X_1,X_2})$

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Main results

Causal cribbing

Theorem

A source (U, V) is transmissible via the MAC $P_{Y|X_1,X_2}$ with causal cribbing, at rate ρ, if and only if

$$\rho H(U|V) \leq H(X_1)$$
$$\rho H(V|U) \leq I(X_2; Y|X_1)$$
$$\rho H(U, V) \leq I(X_1, X_2; Y)$$

for some P_{X_1,X_2}.

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Causal cribbing

Theorem

A source (U, V) *is transmissible via the MAC* $P_{Y|X_1,X_2}$ *with causal cribbing, at rate* ρ*, if and only if*

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for some P_{X_1,X_2}.

$\Longrightarrow$ Separation?

Main results

Causal cribbing - discussion

Assume the conditions are satisfied. Then

$$\begin{aligned} \rho H(U|V) &\leq H(X_1) \\ \rho H(V|U) &\leq I(X_2; Y|X_1) \\ \rho H(U,V) &\leq I(X_1, X_2; Y) \end{aligned}$$

Main results

Causal cribbing - discussion

Assume the conditions are satisfied. Then

$$\rho H(U|V) \leq R_1 \leq H(X_1)$$
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But for separation to hold we need

$$\rho H(U, V) \leq R_1 + R_2 \leq I(X_1, X_2; Y)$$

Main results

Causal cribbing - discussion

- $\mathcal{I}_1 = [a, b]$, $\mathcal{I}_2 = [c, d]$, $\mathcal{I}_3 = [e, f]$ nonempty intervals.
- Necessary and sufficient conditions for the existence of (R_1, R_2) s.t.

$$R_1 \in \mathcal{I}_1, \quad R_2 \in \mathcal{I}_2, \quad R_1 + R_2 \in \mathcal{I}_3$$

is that the endpoints satisfy

$$a + c \leq f, \quad b + d \geq e$$

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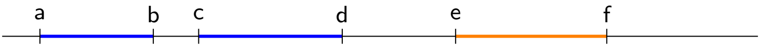

Identify the intervals

$$\mathcal{I}_1 = \left[\rho H\left(U|V\right), H\left(X_1\right)\right]$$
$$\mathcal{I}_2 = \left[\rho H\left(V|U\right), I\left(X_2; Y|X_1\right)\right]$$
$$\mathcal{I}_3 = \left[\rho H\left(U, V\right), I\left(X_1, X_2; Y\right)\right]$$

By basic properties of information functions we have

$$\begin{aligned}
\rho H(U|V) + \rho H(V|U) &\leq \rho H(U, V) \\
&\leq I(X_1, X_2; Y) \\
&= I(X_1; Y) + I(X_2; Y|X_1) \\
&\leq H(X_1) + I(X_2; Y|X_1)
\end{aligned}$$

Implying that we can find

$$\rho H(U, V) \leq R_1 + R_2 \leq I(X_1, X_2; Y)$$

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Theorem

A source (U, V) is transmissible via the MAC $P_{Y|X_1,X_2}$ with causal cribbing, at rate ρ, if and only if

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- The source pair is transmissible via the MAC at rate ρ, with causal cribbing, if and only if its (scaled) SW region intersects the MAC capacity region with causal cribbing.

A separation principle applies

Main results

Strictly causal cribbing

Theorem

A source (U, V) is transmissible via the MAC $P_{Y|X_1,X_2}$ with strictly causal cribbing, at rate $\rho = 1$, if

$$H(U|V) \leq H(X_1|V, W)$$
$$H(V|U) \leq I(X_2; Y|X_1, U, W)$$
$$H(U, V) \leq I(X_1, X_2, Y|U)$$

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$$P_W P_{U,V} P_{X_1|U,W} P_{X_2|V,W}$$

where $P_W = P_U$.

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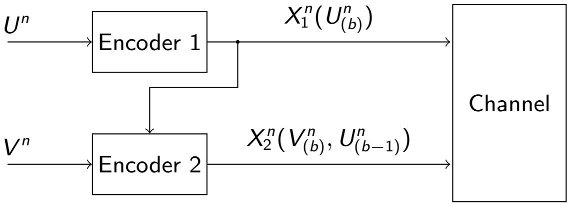

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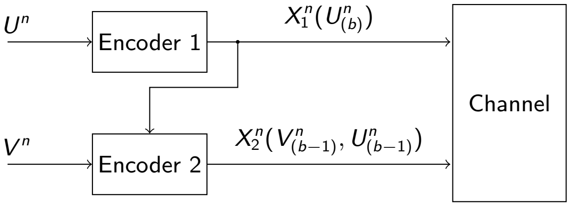

Future work:

- Other cribbing models (two sided, non causal)
- Transmission with distortion

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