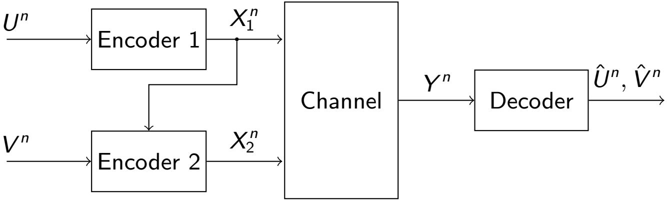
Joint Source-Channel Coding for Cribbing Models

Eliron Amir and Yossef Steinberg

ISIT 2012

Problem formulation

One sided cribbing



- Encoders:

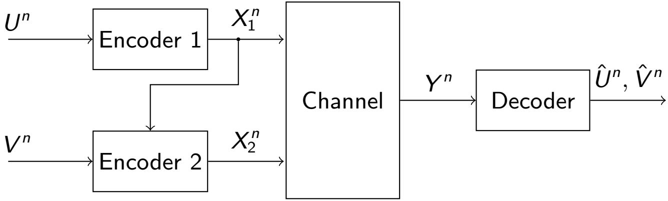
$$X_{1,i} = f_{1,i}(U^n)$$
$$X_{2,i} = f_{2,i}(V^n, X_1^{i-1}) \quad \text{(strictly causal cribbing)},$$

- Memoryless channel $P_{Y|X_1,X_2}$ and source $(U, V) \sim P_{U,V}$
- Lossless/Lossy transmission of U^n and V^n.

Transmissibility conditions - ?

Problem formulation

One sided cribbing



- Encoders:

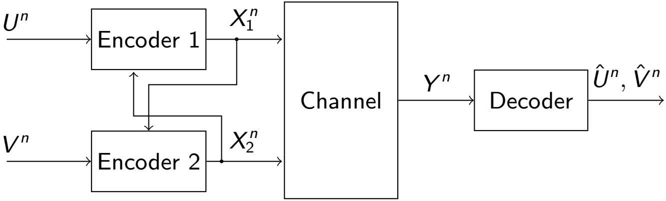
$$X_{1,i} = f_{1,i}(U^n)$$
$$X_{2,i} = f_{2,i}(V^n, X_1^i) \quad \text{(causal cribbing)}$$

- Memoryless channel $P_{Y|X_1,X_2}$ and source $(U,V) \sim P_{U,V}$
- Lossless/Lossy transmission of U^n and V^n.

Transmissibility conditions - ?

Problem formulation

Two sided cribbing



We study also models with strictly causal cribbing on both sides:

$$X_{1,i} = f_{1,i}(U^n, X_2^{i-1}), \quad X_{2,i} = f_{2,i}(U^n, X_1^{i-1}),$$

and strictly causal / causal cribbing:

$$X_{1,i} = f_{1,i}(U^n, X_2^{i-1}), \quad X_{2,i} = f_{2,i}(U^n, X_1^{i}).$$

Related work

Channel coding, source coding:

- Willems & van der Meulen, 85: Capacity of MAC with cribbing encoders

Related work

Channel coding, source coding:

- Willems & van der Meulen, 85: Capacity of MAC with cribbing encoders
- Slepian & Wolf, July 73: Noiseless coding of correlated sources

Related work

Channel coding, source coding:

- Willems & van der Meulen, 85: Capacity of MAC with cribbing encoders
- Slepian & Wolf, July 73: Noiseless coding of correlated sources
- Lossy S&W - Still open. Achievability - Berger & Tung 77. Solved for one sided distortion by Berger & Yeung 89.

Related work

Channel coding, source coding:

- Willems & van der Meulen, 85: Capacity of MAC with cribbing encoders
- Slepian & Wolf, July 73: Noiseless coding of correlated sources
- Lossy S&W - Still open. Achievability - Berger & Tung 77. Solved for one sided distortion by Berger & Yeung 89.

Joint S-C coding:

Related work

Channel coding, source coding:

- Willems & van der Meulen, 85: Capacity of MAC with cribbing encoders
- Slepian & Wolf, July 73: Noiseless coding of correlated sources
- Lossy S&W - Still open. Achievability - Berger & Tung 77. Solved for one sided distortion by Berger & Yeung 89.

Joint S-C coding:

- Slepian & Wolf, Sept. 73: Multiple access channel with correlated sources:
  - Two users MAC, three independent sources (U_0, U_1, U_2), U_0 known to both users.

Related work

Channel coding, source coding:

- Willems & van der Meulen, 85: Capacity of MAC with cribbing encoders
- Slepian & Wolf, July 73: Noiseless coding of correlated sources
- Lossy S&W - Still open. Achievability - Berger & Tung 77. Solved for one sided distortion by Berger & Yeung 89.

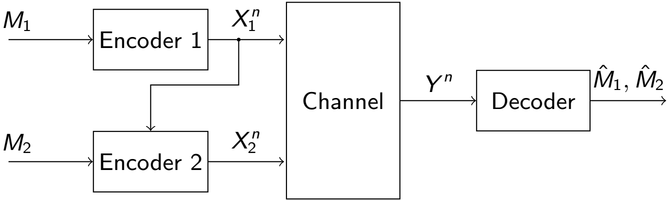
Joint S-C coding:

- Slepian & Wolf, Sept. 73: Multiple access channel with correlated sources:
  - Two users MAC, three independent sources (U_0, U_1, U_2), U_0 known to both users.
- Cover, El Gamal, & Salehi, 80: Multiple access channels with arbitrarily correlated sources. Sufficient conditions for transmissibility. Separation does not hold. Not optimal (Dueck 81).

Related work

Capacity of MAC with cribbing encoders, W&M, 85:

W&M studied all possible cribbing combinations:



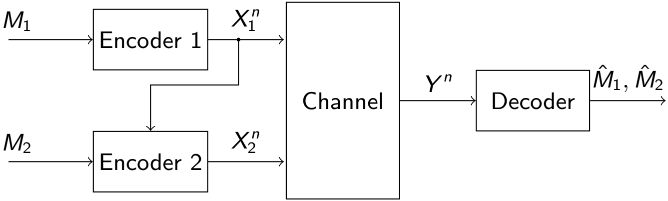
One sided, strictly causal cribbing

$$X_{1,i} = f_{1,i}(M_1)$$
$$X_{2,i} = f_{2,i}(M_2, X_1^{i-1})$$

Related work

Capacity of MAC with cribbing encoders, W&M, 85:

W&M studied all possible cribbing combinations:



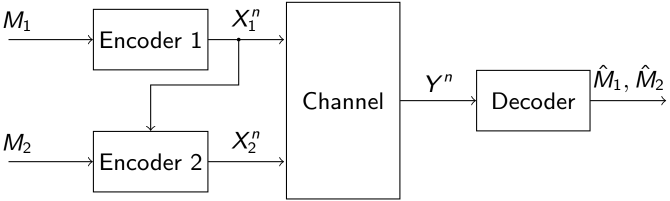
One sided, causal cribbing

$$X_{1,i} = f_{1,i}(M_1)$$
$$X_{2,i} = f_{2,i}(M_2, X_1^i)$$

Related work

Capacity of MAC with cribbing encoders, W&M, 85:

W&M studied all possible cribbing combinations:



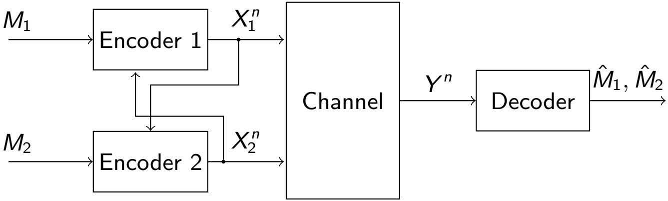
One sided, non-causal cribbing

$$X_{1,i} = f_{1,i}(M_1)$$
$$X_{2,i} = f_{2,i}(M_2, X_1^n)$$

Related work

Capacity of MAC with cribbing encoders, W&M, 85:

W&M studied all possible cribbing combinations:



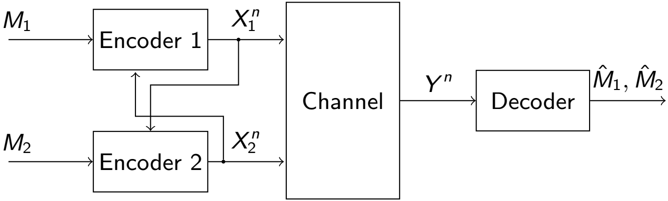
Two sided, stricly causal

$$X_{1,i} = f_{1,i}(M_1, X_2^{i-1})$$
$$X_{2,i} = f_{2,i}(M_2, X_1^{i-1})$$

Related work

Capacity of MAC with cribbing encoders, W&M, 85:

W&M studied all possible cribbing combinations:



Two sided, causal

$$X_{1,i} = f_{1,i}(M_1, X_2^{i-1})$$
$$X_{2,i} = f_{2,i}(M_2, X_1^{i})$$

Related work

Capacity of MAC with cribbing encoder, W&M 85

Main theme: cribbing allows dependence between the MAC inputs:

Related work

Capacity of MAC with cribbing encoder, W&M 85

Main theme: cribbing allows dependence between the MAC inputs:

- The capacity region for one sided strictly causal cribbing

$$\begin{aligned} R_1 &\leq H(X_1|W) \\ R_2 &\leq I(X_2; Y|X_1, W) \\ R_1 + R_2 &\leq I(X_1, X_2; Y) \end{aligned}$$

for some $P_W P_{X_1|W} P_{X_2|W}$.

Related work

Capacity of MAC with cribbing encoder, W&M 85

Main theme: cribbing allows dependence between the MAC inputs:

- The capacity region for one sided strictly causal cribbing

$$
\begin{aligned}
R_1 &\le H(X_1|W) \\
R_2 &\le I(X_2;Y|X_1,W) \\
R_1 + R_2 &\le I(X_1,X_2;Y)
\end{aligned}
$$

  for some $P_W P_{X_1|W} P_{X_2|W}$.

- The capacity region for one sided causal/non-causal cribbing

$$
\begin{aligned}
R_1 &\le H(X_1) \\
R_2 &\le I(X_2;Y|X_1) \\
R_1 + R_2 &\le I(X_1,X_2;Y)
\end{aligned}
$$

  for some P_{X_1,X_2}.

Related work

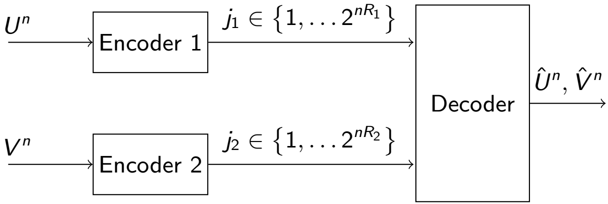
MAC without cribbing

$$\begin{aligned} R_1 &\leq I(X_1; Y|X_2, Q) \\ R_2 &\leq I(X_2; Y|X_1, Q) \\ R_1 + R_2 &\leq I(X_1, X_2; Y|Q) \end{aligned}$$

for some $P_Q P_{X_1|Q} P_{X_2|Q}$.

Related work

Noisless coding of correlated sources, S&W 73

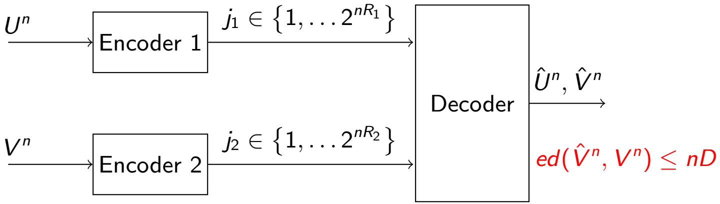


A rate pair (R_1, R_2) is achievable if and only if

$$\begin{aligned} R_1 &\geq H(U|V) \\ R_2 &\geq H(V|U) \\ R_1 + R_2 &\geq H(U, V) \end{aligned}$$

Related work

S&W with one sided distortion, Berger & Yeung 89



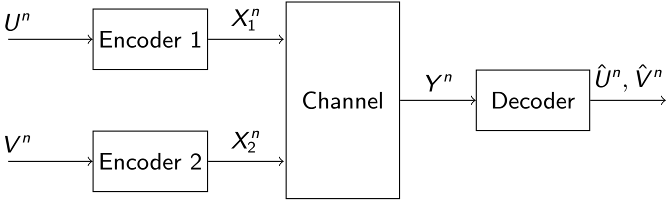
A rate-distortion triple (R_1, R_2, D) is achievable if and only if

$$\begin{aligned} R_1 &\geq H(U|W) \\ R_2 &\geq I(V;W|U) \\ R_1 + R_2 &\geq H(U) + I(V;W|U) \\ D &\geq Ed(\phi(U,W), V) \end{aligned}$$

for some $\phi(U, W)$ and external rv W, $U - V - W$.

Related work

Joint source-channel coding for MAC, Cover, El Gamal, & Salehi 80



The source (U, V) can be sent via the MAC with $P_e \to 0$ if

$$\begin{aligned} H(U|V) &\leq I(X_1; Y|X_2, V, W) \\ H(V|U) &\leq I(X_2; Y|X_1, U, W) \\ H(U, V|S) &\leq I(X_1, X_2; Y|W, S) \\ H(U, V) &\leq I(X_1, X_2; Y) \end{aligned}$$

where S is the *common part* $g(U) = f(V) = S$, and

$$P_{W,U,V,X_1,X_2} = P_W P_{U,V} P_{X_1|U,W} P_{X_2|V,W}.$$

Related work

Joint source-channel coding for MAC, Cover, El Gamal, & Salehi 80

Without common part:

The source (U, V) can be sent via the MAC with $P_e \to 0$ if

$$\begin{aligned} H(U|V) &\leq I(X_1; Y|X_2, V) \\ H(V|U) &\leq I(X_2; Y|X_1, U) \\ H(U, V) &\leq I(X_1, X_2; Y) \end{aligned}$$

where

$$P_{U,V,X_1,X_2} = P_{U,V} P_{X_1|U} P_{X_2|V}.$$

Related work

Joint source-channel coding for MAC, Cover, El Gamal, & Salehi 80

Without common part:

The source (U, V) can be sent via the MAC with $P_e \to 0$ if

$$\begin{aligned} H(U|V) &\le I(X_1; Y|X_2, V) \\ H(V|U) &\le I(X_2; Y|X_1, U) \\ H(U, V) &\le I(X_1, X_2; Y) \end{aligned}$$

where

$$P_{U,V,X_1,X_2} = P_{U,V} P_{X_1|U} P_{X_2|V}.$$

- Separation does not hold (example, + r.h.s depends on the source)

Related work

Joint source-channel coding for MAC, Cover, El Gamal, & Salehi 80

Without common part:

The source (U, V) can be sent via the MAC with $P_e \to 0$ if

$$\begin{aligned} H(U|V) &\leq I(X_1; Y|X_2, V) \\ H(V|U) &\leq I(X_2; Y|X_1, U) \\ H(U, V) &\leq I(X_1, X_2; Y) \end{aligned}$$

where

$$P_{U,V,X_1,X_2} = P_{U,V} P_{X_1|U} P_{X_2|V}.$$

- Separation does not hold (example, + r.h.s depends on the source)
- Sub optimal (Dueck 81)

Related work

Joint source-channel coding for MAC, Cover, El Gamal, & Salehi 80

- An intuitive explanation: in channel capacity problems, the messages at the inputs, M_1, M_2 are independent, resulting in $P_{X_1,X_2} = P_{X_1} P_{X_2}$. This does not fit well the source model - U and V are dependent. (??)
- Cribbing allows dependence between inputs. Does separation yield optimal performance in cribbing models?

Definitions

Strictly causal one sided cribbing, lossless transmission

- An (n, m, ϵ) code consists of $m + 1$ encoding functions

$$f_1 : \mathcal{U}^n \to \mathcal{X}_1^m,$$

$$f_{2,i} : \mathcal{V}^n \times \mathcal{X}_1^{i-1} \to \mathcal{X}_{2,i}, \quad i = 1, 2, \ldots, m$$

  and a decoding function

$$\phi : \mathcal{Y}^m \to \mathcal{U}^n \times \mathcal{V}^n$$

  such that $P((U^n, V^n) \neq \phi(Y^m)) \leq \epsilon$.

- The rate of the code is $\rho = n/m$

Definitions

Strictly causal cribbing

(U, V) is transmissible via $P_{Y|X_1,X_2}$ at rate ρ if for every $\epsilon > 0$, $\delta > 0$, and s.l. n, there exists an $(n, n/(\rho - \delta), \epsilon)$ code for $((U, V), P_{Y|X_1,X_2})$

Definitions

Causal cribbing

- An (n, m, ϵ) code consists of $m + 1$ encoding functions

$$f_1 : \mathcal{U}^n \to \mathcal{X}_1^m,$$

$$f_{2,i} : \mathcal{V}^n \times \mathcal{X}_1^i \to \mathcal{X}_{2,i}, \quad i = 1, 2, \ldots, m$$

  and a decoding function

$$\phi : \mathcal{Y}^m \to \mathcal{U}^n \times \mathcal{V}^n$$

  such that $P((U^n, V^n) \neq \phi(Y^m)) \leq \epsilon$.

- The rate of the code is $\rho = n/m$

Definitions

Causal cribbing with one sided distortion

An (n, m, D, ϵ) joint source channel code with causal cribbing by encoder 2 consists of $m + 1$ encoding functions

$$f_1 : \mathcal{U}^n \to \mathcal{X}_1^m,$$

$$f_{2,i} : \mathcal{V}^n \times \mathcal{X}_1^i \to \mathcal{X}_{2,i}, \quad i = 1, 2, \ldots, m$$

Definitions

Causal cribbing with one sided distortion

An (n, m, D, ϵ) joint source channel code with causal cribbing by encoder 2 consists of $m+1$ encoding functions

$$f_1 : \mathcal{U}^n \to \mathcal{X}_1^m,$$

$$f_{2,i} : \mathcal{V}^n \times \mathcal{X}_1^i \to \mathcal{X}_{2,i}, \quad i = 1, 2, \ldots, m$$

and a pair of decoders

$$d_u^m : \mathcal{Y}^m \to \mathcal{U}^n$$
$$d_v^m : \mathcal{Y}^m \to \hat{\mathcal{V}}^n$$

such that the probability of error in decoding U does not exceed ϵ:

$$P_e^{(u)} = Pr\left\{U^n \neq d_u^m\left(Y^m\right)\right\} \leq \epsilon.$$

and the average distortion in decoding V is at most D:

$$\mathsf{E}d(V^n, d_v^m(Y^m)) \leq nD$$

Main results overview

One sided cribbing:

Main results overview

One sided cribbing:

- One sided strictly causal cribbing, lossless transmission (presented in IZS 2012)

Main results overview

One sided cribbing:

- One sided strictly causal cribbing, lossless transmission (presented in IZS 2012)
- One sided causal cribbing, lossless transmission (presented in IZS 2012)

Main results overview

One sided cribbing:

- One sided strictly causal cribbing, lossless transmission (presented in IZS 2012)
- One sided causal cribbing, lossless transmission (presented in IZS 2012)
- One sided causal/noncausal cribbing with one sided distortion

Main results overview

One sided cribbing:

- One sided strictly causal cribbing, lossless transmission (presented in IZS 2012)
- One sided causal cribbing, lossless transmission (presented in IZS 2012)
- One sided causal/noncausal cribbing with one sided distortion

Two sided cribbing:

Main results overview

One sided cribbing:

- One sided strictly causal cribbing, lossless transmission (presented in IZS 2012)
- One sided causal cribbing, lossless transmission (presented in IZS 2012)
- One sided causal/noncausal cribbing with one sided distortion

Two sided cribbing:

- Strictly causal cribbing by encoder 1 and causal cribbing by encoder 2

Main results overview

One sided cribbing:

- One sided strictly causal cribbing, lossless transmission (presented in IZS 2012)
- One sided causal cribbing, lossless transmission (presented in IZS 2012)
- One sided causal/noncausal cribbing with one sided distortion

Two sided cribbing:

- Strictly causal cribbing by encoder 1 and causal cribbing by encoder 2
- Strictly causal cribbing by both encoders

Main results

One sided causal/noncausal cribbing, one sided distortion

Theorem

(U, V) can be sent with arbitrarily small probability of error for U and distortion D for V over $P_{Y|X_1,X_2}$, with causal or non-causal cribbing by Encoder 2 at rate $\rho = 1$ if and only if

$$H(U \mid V) \leq H(X_1) \quad (1)$$

$$I(V; W \mid U) \leq I(Y; X_2 \mid X_1) \quad (2)$$

$$H(U) + I(V; W \mid U) \leq I(Y; X_1, X_2) \quad (3)$$

$$Ed(V, W) \leq D \quad (4)$$

for some

$$P_{U,V} P_{W|U,V} P_{X_1,X_2} P_{Y|X_1,X_2}.$$

Note: W is the reconstruction.

Main results

One sided causal/noncausal cribbing

Q: Do we have separation?

Main results

One sided causal/noncausal cribbing

Q: Do we have separation?

A: Only from code design perspective. Operatively, encoder 2 must decode encoder's 1 message in order to choose the appropriate compressed word W^n.

The left hand side is not the Berger-Yeung region.

Main results

One sided causal cribbing - no distortion (ISZ 2012)

Corollary

A source (U, V) *is transmissible via the MAC* $P_{Y|X_1,X_2}$ *with one sided causal cribbing, at rate* ρ*, if and only if*

$$\rho H(U|V) \leq H(X_1)$$
$$\rho H(V|U) \leq I(X_2; Y|X_1)$$
$$\rho H(U, V) \leq I(X_1, X_2; Y)$$

for some P_{X_1,X_2}*.*

Main results

One sided causal cribbing - no distortion (ISZ 2012)

Corollary

A source (U, V) is transmissible via the MAC $P_{Y|X_1,X_2}$ with one sided causal cribbing, at rate ρ, if and only if

$$\rho H(U|V) \leq H(X_1)$$
$$\rho H(V|U) \leq I(X_2; Y|X_1)$$
$$\rho H(U, V) \leq I(X_1, X_2; Y)$$

for some P_{X_1,X_2}.

- The source pair is transmissible via the MAC at rate ρ, with causal cribbing, if and only if its (scaled) SW region intersects the MAC capacity region with causal cribbing.

A separation principle applies

Main results

Strictly causal cribbing by encoder 1 + causal cribbing by encoder 2

Theorem

(U, V) can be sent with arbitrarily small probability of error for U and distortion D for V over $P_{Y|X_1,X_2}$, with strictly causal cribbing by Encoder 1 and causal cribbing by Encoder 2 at rate $\rho = 1$ if and only if

$$H(U \mid V) \leq H(X_1) \tag{5}$$

$$I(V; W \mid U) \leq H(X_2 \mid X_1) \tag{6}$$

$$H(U) + I(V; W \mid U) \leq I(Y; X_1, X_2) \tag{7}$$

$$Ed(V, W) \leq D \tag{8}$$

for some

$$P_{U,V} P_{W|U,V} P_{X_1,X_2} P_{Y|X_1,X_2}$$

Main results

One sided strictly causal cribbing (IZS 2012)

Theorem

A source (U, V) *is transmissible via the MAC* $P_{Y|X_1,X_2}$ *with one sided strictly causal cribbing, at rate* $\rho = 1$*, if*

$$\begin{aligned} H(U|V) &\leq H(X_1|V, W) \\ H(V|U) &\leq I(X_2; Y|X_1, U, W) \\ H(U, V) &\leq I(X_1, X_2, Y|U) \end{aligned}$$

for some

$$P_W P_{U,V} P_{X_1|U,W} P_{X_2|V,W}$$

Main results

One sided strictly causal cribbing (IZS 2012)

Theorem

A source (U, V) is transmissible via the MAC $P_{Y|X_1,X_2}$ with one sided strictly causal cribbing, at rate $\rho = 1$, if

$$H(U|V) \leq H(X_1|V, W)$$
$$H(V|U) \leq I(X_2; Y|X_1, U, W)$$
$$H(U, V) \leq I(X_1, X_2, Y|U)$$

for some

$$P_W P_{U,V} P_{X_1|U,W} P_{X_2|V,W}$$

- Separation does not hold

Main results

Two Sided Strictly Causal Cribbing

Theorem

A source (U, V) *is transmissible via the MAC* $P_{Y|X_1,X_2}$ *with two sided strictly causal cribbing, at rate* $\rho = 1$*, if*

$$H(U|V) \leq H(X_1|V, W)$$
$$H(V|U) \leq H(X_2|U, W)$$
$$H(U, V) \leq I(X_1, X_2, Y|U, V)$$

for some

$$P_W P_{U,V} P_{X_1|U,W} P_{X_2|V,W}$$

Main results

Two Sided Strictly Causal Cribbing

Theorem

A source (U, V) is transmissible via the MAC $P_{Y|X_1,X_2}$ with two sided strictly causal cribbing, at rate $\rho = 1$, if

$$H(U|V) \le H(X_1|V, W)$$
$$H(V|U) \le H(X_2|U, W)$$
$$H(U, V) \le I(X_1, X_2, Y|U, V)$$

for some

$$P_W P_{U,V} P_{X_1|U,W} P_{X_2|V,W}$$

- Separation does not hold

Future work:

- Close the gap for strictly causal models.
- Find examples for situations were separation is not optimal.